our organizations is perhaps the greatest justification for their existence. Let us encourage the members of our teaching staffs to make an extra effort to attend our meetings and in this, as in other ways, to share actively in promoting the helpfulness of these gatherings. In particular, let us plan definitely to take part in the September meetings at Ohio State University.

The sessions of the American Association for the Advancement of Science began with the formal opening on Monday evening in the Auditorium of the Junior College, Dr. J. McKeen Cattell, editor of *Science*, giving the retiring presidential address on "Some psychological experiments." This address appeared in *Science* for January 1 and 8. The Kansas City meetings were noteworthy for the large number of lectures which were offered after the pattern of the British Association lectures, popular presentations of recent advances in science given by outstanding men in various fields. On Tuesday afternoon Professor D. C. Miller of the Case School of Applied Science delivered an address as president of the American Physical Society on "The Michelson-Morley ether-drift experiment, its history and significance"; he gave the finished results of his experiments on the repetition and refinement of the Michelson-Morley work, which he has been conducting on Mount Wilson for the past two years, and reported his conclusion, on the basis of extensive computations, that the solar system is moving toward the constellation of the Dragon with a velocity of 200 kilometers per second, the earth dragging the ether along with it to the extent of 95 per cent with a residual "slip" of 5 per cent. This address was regarded as a notable contribution to the advancement of science by the American Association committee on award and as such was awarded the American Association Prize of one thousand dollars.

On Tuesday evening Professor R. A. Millikan, director of the Norman Bridge Physical Laboratory, lectured on "The stripped atom." On Wednesday afternoon the American Association paid the American Mathematical Society the compliment of making the third annual Josiah Willard Gibbs lecture a general session of the American Association. Before an audience of 600 or more the lecture was delivered by Professor James Pierpont on "The history of man's effort to solve the problem of space, and the effect of relativity on our views"; this address will be published in the *Bulletin*. At the same hour in another auditorium four lectures were given under the auspices of the American Association's special committee, on the place of the sciences in education; these papers were by Dr. O. W. Caldwell, of the Lincoln School, Columbia University, chairman of the special committee, Dr. B. E. Livingston, Johns Hopkins University; Dr. E. E. Slosson, of Science Service, and Professor M. I. Pupin, Columbia University, president of the Association. Among the other general sessions and addresses may be mentioned a session on the relations of engineering to the fundamental sciences, a session under the auspices of

the committee of one hundred on scientific research, and a lecture on Thursday evening by Professor F. R. MOULTON, University of Chicago, on "The origin and evolution of worlds." Full accounts of the American Association meetings and numerous papers on the programs will appear in early issues of *Science*.

Professor L. H. BAILEY of Cornell University was elected president of the American Association for 1926. On nomination by the section committee for Section A, Professor E. V. HUNTINGTON was chosen vice-president and Professor H. H. MITCHELL member of the section committee, Professor R. C. ARCHIBALD continuing his term as secretary.

The joint dinner of the mathematicians with 127 persons present was held at the City Club on Wednesday evening. Professor F. R. Moulton introduced the speakers in a happy fashion. Professor Hedrick told of the transfer of the printing of the *Bulletin* and *Transactions* to this country, of the subsidizing of these journals by a grant obtained through the National Academy of Sciences, and of the promise of excellent work which is shown at the outset of this new venture. Professor Pierpont gave delightful reminiscences of Josiah Willard Gibbs, under whom he had studied at Yale. Dr. Jewell Hughes recalled graduate days at the University of Chicago and various courses under Professor Moulton. Professor Coolidge as the representative of the Mathematical Association spoke of the pleasant experiences at Kansas City, of the recognition by the American Association of mathematics as one of the sciences, adding with true eloquence a tribute to mathematics as an art, to which we are drawn by its beauty, by its cultural and ideal spirit and by the symbolic character of mathematics which it shares with the other arts. Professor Carmichael discussed the possibility of mathematical meetings at Nashville in Dec. 1927 in affiliation with the American Association; he expressed his judgment that there is promise enough in such a meeting to make it profitable and pleasant to all concerned. Various speakers followed with expressions of agreement, of the feasibility of meeting at Nashville which is not so far distant as are other of our meeting places, and of the desirability of holding mathematical meetings as far south as possible for the benefit and inspiration of teachers in the South. Those present at the dinner voted unanimously in favor of holding the meetings at Nashville. At the conclusion of this dinner the thanks of Section A, the Mathematical Association, and the Mathematical Society were voted especially to Professors Luby and Stouffer for the excellent preparations and the delightful appointments for the dinner. By formal vote at one of the sessions, the cordial greetings of the mathematicians at Kansas City were sent to their colleagues assembled at Hunter College.

In a joint meeting of Section A with Section L (Historical and Philological Sciences) Professor W. H. ROEVER of Washington University read an interesting paper on "William Chauvenet and his mathematical contribution to astronomy."

The American Mathematical Society held its twenty-fourth Western meeting together with that of the Southwestern Section on Tuesday and Wednesday. Forty-four papers were read at the two sessions on Tuesday. The third Gibbs lecture has already been noted above.

The meetings of the Mathematical Association consisted of a joint session on Wednesday morning and two separate sessions on Thursday. The program was prepared under the following Committee: L. W. DOWLING, chairman; W. C. BRENKE, OLIVE C. HAZLETT, G. H. LIGHT. This association joined with the other two organizations on Wednesday morning, but without a formal representative, in order that there might be adequate time for the two important addresses of that session. Professor H. J. Ettlinger, who was to have given a paper on "Mathematical clubs for colleges and universities," was prevented from attending the meetings through sickness in the family, and the committee decided to postpone this paper to a later meeting at which it might be presented by the author. Abstracts of some of the papers are given, numbered in accordance with the numbers of the papers.

JOINT SESSION OF THE ASSOCIATION WITH THE AMERICAN MATHEMATICAL SOCIETY AND SECTION A OF THE AMERICAN ASSOCIATION

(1) "The Heine-Borel theorem and allied problems" by Professor T. H. HILDEBRANDT, University of Michigan, Vice-president of the American Mathematical Society.

(2) "The algebraic numbers and division" by Professor J. C. FIELDS, University of Toronto, retiring vice-president of Section A.

1. Professor Hildebrandt's address will appear in the *Bulletin* of the American Mathematical Society.

2. Professor Fields' paper will appear in *Science*.

FIRST SESSION OF THE ASSOCIATION.

(3) "Robert Adrain and the beginnings of American mathematics,"—retiring presidential address by Professor J. L. COOLIDGE, Harvard University.

(4) "The definition of function and its effect on elementary and advanced instruction" by Professor E. R. HEDRICK, University of California, Southern Branch.

3. President Coolidge's paper appeared in full in the February issue.

4. Professor Hedrick reviewed the elementary definition of function in its classical sense, and certain closely associated concepts. The modern extensions of these concepts were stated and the effect of the concept and its associated ideas on both elementary and advanced instruction was emphasized, with a considerable number of specific instances. It was shown that the content of courses in mathematics, and the method of instruction, would be greatly affected by realization, on the part of teachers, of the fundamental connections which exist with the function concept.

SECOND SESSION OF THE ASSOCIATION

(5) "Determinants and their principal minors" by Professor E. B. STOUFFER, University of Kansas.

(6) "The course in statistics in the department of mathematics," by Professor A. R. CRATHORNE, University of Illinois.

(7) "Some applications of mathematics to architecture" by Professor E. C. PHILLIPS, Georgetown University.

(8) "A new method of determining a series solution of linear differential equations with constant or variable coefficients" by Mr. W. O. PENNELL, Chief engineer, Southwestern Bell Telephone Company, St. Louis.

5. Professor Stouffer indicated briefly the work which has been done in discovering relations among the principal minors of determinants. He pointed out that a certain theorem by McMahon published in 1893 is incorrect. This theorem was apparently fundamental in the study of the relations among the principal minors, and as a consequence of its acceptance by later writers, only results of little importance have been obtained. In the present paper McMahon's error is pointed out, and complete systems of principal minors for the general determinant are derived and also complete systems of relations among the principal minors are obtained. The whole idea is generalized to apply to certain combinations of any number of determinants. Several applications of the results to other fields of mathematics are indicated.

6. Professor Crathorne's paper appears in full in this number of the MONTHLY.

7. After a brief historical account of the origin and development of Gothic architecture and a description of the three main styles of decorative tracery used especially in the windows of Gothic buildings, namely geometrical tracery, flowing tracery and flamboyant tracery, Professor Phillips gave a simple mathematical expression for the general groundwork of all such tracery which resembles the outlines of a flat flower with pointed petals; from this simple expression which represents very well the earliest forms of flowing tracery there were obtained, by suitable modifications of the arbitrary constants of the equations, the mathematical expressions to represent the somewhat more complex forms which architects introduced as the art of Gothic decoration advanced in perfection. A method was then given for changing the shape of the petal-like elements of the decorative scheme without altering the mutual relations of these elements among themselves; the varying artistic effect thus produced in the design was studied and an attempt was made to find out which of these forms was most pleasing. The last portion of the paper treated briefly the mathematical equations representing the latest, and probably the least perfect, type of tracery, the one which was used in flamboyant gothic.

Mathematically the entire paper may be summed up in the brief statement that all forms of Gothic window tracery are fairly well represented by the equations:

$$\theta=(c/p)(t-\tfrac{1}{2}\sin 2t)+b\rho\,,$$
$$\rho=1+k\cos t-m\sin^2 t\cos t\,,$$

where θ and ρ are polar coördinates, t is the parametric variable, and c, p, b, k and m are arbitrary constants.

8. The method described by Mr. Pennell is an operational method for the solution of any linear differential equation. Heretofore operational methods have been developed for the solution of linear equations with constant coefficients. The method outlined in this paper is applicable to any linear equation with constant or with variable coefficients. The method is one of operational division. The differential equation is first rewritten substituting for the differential coefficients d^n/dx^n the operational symbol p^n. The equation is then solved by algebraic methods for the unknown quantity which is expressed in terms of p and the independent variable. This is called the operational solution. The series solution is then obtained from the operational solution by a method of operational long division. This method is the key to the entire process and reduces the series solution of any linear equation to what is substantially an algebraic process. For many equations it is believed that this gives a much more simple and direct solution than any hitherto known.

The theory was developed by showing that there is an algebra which obeys the commutative, distributive and associative laws and with one exception parallels in its operation the results obtained by integration and differentiation so that the operational division by which the solution of the differential equations is obtained is merely long division following the rules of the operational algebra.

Meeting of the Board of Trustees of the Association.

Nine members of the Board were present at the various sessions.

The following twenty-nine persons, on applications duly certified, were elected to individual membership:

Violet M. Andrews, Junior Student, Oberlin Coll., Oberlin, Ohio.

J. W. Blincoe, A.M. (Virginia). Prof., Math. and Physics, Central Wesleyan Coll., Warrenton, Mo.

M. Sue Burney, A.M. (Chicago). Instr., Jr. Coll., St. Joseph, Mo.

W. H. Buxton, A.M. (Oregon). Prof., Whitworth Coll., Spokane, Wash.

Joseph Canning, A.B. (Intermountain Union Coll.). Railroad Mach., Helena, Mont.

D. S. Dearman, M.S. (Millsaps), A.M. (Vanderbilt). Head of Dept., Ky. Wesleyan Coll., Winchester Ky.

Mrs. Flora H. Eaton, A.M. (North Carolina). Registrar and Head of Dept. of Math., Mars Hill Coll.. Mars Hill, N. C.

Frances Gilmore. Breaux Bridge, La.

G. D. Gore, M.S. (Chicago). Instr., S. Dak. State Coll., Brookings, S. Dak.

Margaret E. Harris, A.M. (Teachers Coll., Columbia). Chair of Math., Grenada Coll., Grenada, Miss.

A. H. Jekel, A.B. (Michigan). Actuary's Dept., Mo. State Life Ins. Co., St. Louis, Mo.

Vladimir Karapetoff, C.E., M.M.E. (Russia). Prof., Elec. Eng., Cornell Univ., Ithaca, N. Y.

Mary C. Martin. Teacher, High School, Mount Morris, Ill.

T. A. Martin, A.M. (Ohio Wesleyan; Yale). Head of Dept., Berea Coll., Berea, Ky.

T. W. MOORE, B.S. (Washington and Jefferson). Asst., Yale Univ., New Haven, Conn.
SISTER M. PRUDENTIA MORIN, A.B. (Catholic Univ. of Amer.). Teacher, Coll. of St. Scholastica, Duluth, Minn.
Miss ARRIA MURTO, B.S. (Missouri). Teacher, High School, Carthage, Mo.
E. G. OLDS, A.M. (Pittsburgh). Asst. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa.
D. D. PEELE, A.M. (Chicago). Dean and Prof. of Math., Columbia Coll., Columbia, S. C.
J. M. RICE, E.E. (Lehigh). Instr., Penna. State Coll., State College, Pa.
SUSAN V. RICHMOND, A.B. (Randolph-Macon W. Coll.). Teacher, Western High School, Washington, D. C.
KATHERINE J. RYNO, A.M. (Tennessee). Asso., Lander Coll., Greenwood, S. C.
R. C. STALEY, Instr., State Agric. Coll., Manhattan, Kans.
P. L. STEVENSON, A.M. (Colorado). Prof., Westminster Coll., Salt Lake City, Utah.
EVELYN R. THOMPSON, A.B. (Mt. Holyoke). Teacher, Western High School, Washington, D. C.
C. H. VEHSE, B.S. (Brown). Asst., Brown Univ., Providence, R. I.
P. L. WELTON, A.M. (Michigan). Instr., East High School, Rochester, N. Y.
W. M. WIBLE, A.M. (Indiana). Prof., Intermountain Union Coll., Helena, Mont.
J. B. WINSLOW, A.B. (Toledo Univ.). Instr., Physics and Math., Toledo Univ., Toledo, Ohio.

The following were appointed associate editors of the MONTHLY for the year 1926:

N. H. ANNING	H. J. ETTLINGER	H. W. KUHN
H. E. BUCHANAN	H. S. EVERETT	C. N. MILLS
W. B. CARVER	B. F. FINKEL	F. D. MURNAGHAN
OTTO DUNKEL	TOMLINSON FORT	D. E. SMITH

The Trustees voted to re-appoint the Secretary and Dean T. M. FOCKE of Case School of Applied Science as the representatives of the Association on the Council of the American Association for 1926 and to appoint Professor DUNHAM JACKSON as the representative of the Association on the National Research Council for a three year term, to succeed Professor H. L. RIETZ whose term expires June 30, 1926. They also voted to hold the annual meeting at Nashville in December 1927 in affiliation with the A.A.A.S.

The Trustees acted on a proposition under which regional associations of teachers of mathematics may be affiliated with this association, members of such associations to be admitted to the Mathematical Association without the payment of the customary initiation fee. Fuller details will be announced when mutual arrangements are more completely made.

In recognition of the services of Mr. Curtis C. Carter, of Bluffs, Illinois, in interesting a friend in the work of the Association, leading to the making of her will in favor of the Association, a bequest of large import for the future, the Trustees voted to make Mr. Carter a life member.

The Trustees directed the sending of a letter to Professor David Eugene Smith voicing the sentiments of the Association on the occasion of his retirement from teaching. The letter follows, together with a reply from Professor Smith:

DEAR PROFESSOR SMITH:

The Mathematical Association of America desires to convey to you its congratulations on your completion of twenty-five years of service in Teachers College, and its deep sense of the significance of your service to the Association and to the teaching of mathematics throughout the country.

You have thrown light on the history of our science, not as a dead record of things gone by, but as the heritage of experience which must be our present guide; you have surveyed the life of our time, not as a spectacle apart, but as a progress in which we must all find our proper place; and you have striven with unfailing devotion and with conspicuous success to share with all men in fullest measure the wisdom that you have achieved.

The Association is proud to claim as one of its leaders a man who has personally contributed so much toward the accomplishment of the purposes to which it is dedicated.

For the Trustees of the Association:

DUNHAM JACKSON, President,

W. D. CAIRNS, Secretary-Treasurer.

MY DEAR COLLEAGUES:

I cannot tell you how deeply I appreciate the message which I have received from you through the hands of Professor Jackson, as President, and Professor Cairns, as Secretary-Treasurer. I cannot fully comprehend the fact that today is the last one of my direct association with this college and university. My closing lecture is given this afternoon. The disagreeable sensation of closing my active connection with our students is rendered less acute by such letters as come to me daily from my friends throughout the country.

I wish to say that one of the delightful experiences in my life has been my connection with the Mathematical Association of America, and with the AMERICAN MATHEMATICAL MONTHLY. I hope at some future time that this appreciation will be more fully understood and in a more tangible way than at present.

Yours very sincerely,

DAVID EUGENE SMITH.

THE CHAUVENET PRIZE.

The committee on the award of the first Chauvenet Prize for excellence in mathematical exposition, Professors W. C. GRAUSTEIN, ANNA PELL WHEELER, and E. B. VAN VLECK, chairman, recommended that the award be made to Professor G. A. BLISS of the University of Chicago for his paper on "Algebraic functions and their divisors," published in the *Annals of Mathematics*, volume 26, Numbers 1 and 2, September and December, 1924. The Trustees voted to approve this choice and to thank the members of the committee for their arduous but very valuable efforts. The award was announced at the business meeting and the prize of one hundred dollars, furnished by a member of the Association, was presented to Professor Bliss following the meetings. His comment on the significance of this prize award is worthy of being shared with the membership of the Association:

DEAR PROFESSOR CAIRNS:

I received recently your letter describing the newly established Chauvenet Prize for excellence in mathematical exposition, and announcing the decision of the Trustees. This, and the substantial check which preceded it, have given me much pleasure, as you will well understand, but I should like to add that the award of the prize for the particular paper you mention has had for me an especially encouraging interest. For many years past the so-called "arithmetic method" in the theory of algebraic functions has seemed to me accessible in the literature with a difficulty quite out of proportion to its completeness and beauty, and I have often wished for a presentation of it which would lead with directness to the fundamental results of the algebraic function theory. When the simplifications in my paper finally occurred to me, after a good deal of study at various times, I thought it would be worth while to set them down in print where they might perhaps aid some future inquirer into this unusually interesting mathematical domain. You will readily understand from these remarks, I think, why the discovery of

this paper by your Committee, and their expression of approval of it, have gratified me especially. I congratulate the Association on the inauguration of the Chauvenet Prize, and hope that it may give to many others in the future the same pleasant impetus which it gives to me.

Yours very sincerely,

G. A. BLISS.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION.

The Secretary-Treasurer announced the names of those elected to membership. He reported also the death of the following members:

G. P. ALDRICH, Instructor in mathematics, University of Iowa (October 8, 1925).

W. W. R. BALL, Trinity College, Cambridge, England (April 4, 1925).

A. G. HALL, Registrar, University of Michigan (January 10, 1925).

W. A. HAMILTON, Professor of mathematics, Antioch College. (June 25, 1925).

W. E. HEAL, U. S. Coast and Geodetic Survey, (October 9, 1925).

P. A. LAMBERT, Professor of mathematics, Lehigh University (February 15, 1925).

MARCIA LATHAM, Instructor in mathematics, Hunter College (May 9, 1925).

L. E. LUNN, Superintendent of schools, Heron Lake, Minnesota (June 18, 1924).

MANSFIELD MERRIMAN, Consulting engineer, New York (June 7, 1925).

M. T. PEED, Professor of mathematics, Emory University (August 28, 1925).

C. E. STROMQUIST, Professor of mathematics, University of Wyoming (April 3, 1925).

G. W. SUBLETTE, Consulting and construction engineer, Minneapolis (November 6, 1925).

W. P. YANCEY, Professor of mathematics and physics, St. Ambrose College (March 13, 1925).

The election of officers for the year 1926 was conducted by mail and in person at this meeting, Professor Crathorne and Mr. Pierson acting as tellers, the result of the ballot being as follows:

For President: A. A. Bennett, 225 votes; Dunham Jackson, 342 votes.

For Vice-Presidents: E. L. Dodd, 212 votes; W. B. Ford, 341 votes; Clara E. Smith, 266 votes; J. W. Young, 298 votes.

For additional members of the Board of Trustees, to serve until January 1929: Florian Cajori, 445 votes; A. B. Chace, 170 votes; J. L. Coolidge 320 votes; B. F. Finkel, 241 votes; C. F. Gummer 189 votes; U. G. Mitchell, 170 votes; E. H. Moore, 366 votes; Oswald Veblen 343 votes.

The following were accordingly declared elected:

President: DUNHAM JACKSON, University of Minnesota.

Vice-Presidents: W. B. FORD, University of Michigan; J. W. YOUNG, Dartmouth College.

Additional members of the Board of Trustees: FLORIAN CAJORI, University of California; J. L. COOLIDGE, Harvard University; E. H. MOORE, University of Chicago; OSWALD VEBLEN, Princeton University.

FINANCIAL REPORT OF THE SECRETARY-TREASURER AS OF DATE, DEC. 17, 1925.

RECEIPTS.		
Balance Dec. 16, 1924		$ 6,847.35
1924 indiv. dues	390.60	
1924 instit. dues	28.00	
1925 indiv. dues	6,036.89	
1925 instit. dues	648.40	
1925 subscriptions	616.11	
Initiation fees	332.00	
Advertising	875.00	
Life membership fees	106.14	
Sale copies MONTHLY	34.87	
Sales copies Register	8.00	
Sale reprints	1.43	
Sales first Monograph	1,166.50	
Gift for Chauvenet Prize	100.00	
For *Annals* subscriptions	4.90	
Interest Oberlin Savings Bank	110.27	
Interest Peoples Bkg. Co.	89.80	
Interest Treasury Note	10.94	
Interest Liberty Loan	10.63	
Interest Hardy Fund	120.00	
Total 1925 receipts		10,690.48
Total assets to the end of 1925 business		$17,537.83

EXPENDITURES.		
Publishers' bills (Oct. '24–Oct. '25)		$ 6,885.50
President's office		43.82
Manager's office		19.96
Editor-in-Chief's office		559.02
Committee on Membership		114.25
Joint Committee on Membership		176.09
Printing 1924 Register		550.00
Part expense 1925 Register		101.75
Printing Peirce reprints		98.85
Secretary-Treasurer's office:		
Postage	$231.45	
Bond	5.00	
Safety deposit	4.00	
Office supplies	29.36	
Express, tel., etc.	43.00	
Clerical work	740.50	
Printing	284.10	
Library expense	53.92	
Refund on subscriptions	7.65	
Washington meetings	88.05	
Ithaca meeting	50.00	
Paid copies of MONTHLY	28.35	
		$1,565.38
Annals subvention		300.00
Paid *Annals* subscriptions		7.90
Paid to sections from initiation fees		135.93
Paid to B. F. Finkel int. Hardy Fund		120.00
Carus Monograph honorarium		300.00
Delegate to Vanderbilt Univ. inauguration		25.00
Transfer Monograph income to certif. of deposit		798.75
Paid Open Court Pub. Co. for Monograph sales		4.00
Total expenditures		$11,806.20
Cash on hand		$ 43.25
Checking account		1,028.37
Oberlin Savings Bank account		2,609.02
Peoples Banking Company acct.		1,519.14
Liberty Bond		500.00
U. S. Treasury Note		500.00
Treas. Savings Certs.		1,000.00
Certificate of dep. Chauv. Fd.		100.00
Bank balance Dec. 17, 1925		$ 7,299.78

Total expenditures		11,806.20
Balance to the end of 1925 business		$ 5,731.63
Received on 1926 business		$ 1,568.15
Book balance Dec. 17, 1925		$ 7,299.78

Of the funds on hand, $311.39 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date January 1, 1926; $1,000 is held, as the beginning of a permanent endowment fund, in the form of U. S. Treasury Savings Certificates, the surrender value on Jan. 1, 1926 being $1,072.50; $167.75 belongs to the Carus Monograph Fund; and a certificate of deposit for $100 is held temporarily as the first Chauvenet Prize Fund.

Aside from the above-mentioned funds on hand, the Carus Monograph Fund, increased by sales and interest, amounts at present to $2,120.17.

When the accounts were closed December 17, 1925, in order to furnish the auditing committee a complete record, there remained on the total business for the year 1925 the following items:

Bills Receivable.	
1925 individual dues	$150.00
1925 institutional dues	42.00
Advertising	100.00
Interest Liberty Loan	10.00
Interest Treasury Note	11.00
	$313.00

Bills Payable	
Publishers' bills (Nov.–Dec. 1925)	$1,400.00
President's office	50.00
Manager's office	30.00
Editor-in-Chief's office	90.00
Other editors' postage	20.00
Committee on Membership	50.00
Secretary-Treasurer's office	140.00
Initiation fees due to sections	500.00
Subvention *Annals*	100.00
Printing 1925 Register	550.00
Printing annual ballots, programs, etc.	150.00
Life Membership Fund	311.00
Carus Monograph Fund	167.75
Chauvenet Prize Fund	100.00
	$3,658.75

If to the balance on 1925 business shown in this report, $5,731.63, there be added the bills receivable, $313.00, and there be subtracted the estimated bills payable, $3,658.75, there results an estimated final balance on 1925 business of approximately $2,380 including the endowment fund of $1,000; the corresponding figure one year ago on 1924 business was $2,500. It is hoped that this somewhat lower figure for this year may be bettered by receiving unpaid dues in arrearage; in spite of the importunity of the treasurer, too many of the Association members have not attended to the payment of their dues or arranged with the treasurer concerning these payments.

W. D. Cairns, *Secretary-Treasurer.*

A PROPOSED AMENDMENT TO THE BY-LAWS OF THE MATHEMATICAL ASSOCIATION

At the Kansas City meeting of the trustees of the Mathematical Association of America, it was voted by a preponderating vote to recommend to the Association that the By-Laws be amended so as to provide for a two-year term for president, the president not to be eligible for re-election. Article III, Section 3 would then read: "The President shall be elected by the Association's members biennially for a term of two years and shall be ineligible for re-election. The Vice-Presidents shall be elected by the Association's members annually etc." As there was a diversity of views on this question, it was voted to present this for the consideration of the members of the Association with a concise statement of the arguments for each side, the amendment to be voted on at the summer meeting in September, 1926. The following two statements have been prepared by two officers of the Association.

Affirmative. "The time has passed when the presidency should be used as a means of honoring members of the Association. The essential thing is that the president should serve the Association, not that the Association should honor the president. Experience has shown that a one year term is too short for any president really to serve as he should. Several of our presidents have expressed the view that they would have liked to do more but the period was too short. The experience of the American Mathematical Society is the same, certain recent presidents having rendered the Society notable service: for example Professor Veblen in raising the endowment and Professor Bliss in enlarging the membership; neither of these services could have been performed by a one year president.

This same result could not be reached by electing the president of the Association for one year, with the possibility of re-election. Since nominations are made in September, it would be quite impossible for the bulk of the membership to appreciate at that time whether or not the president elected at the beginning of the current year was, or was not, particularly capable and interested in his duties. A president would not care to undertake a two year task when he was entirely uncertain as to whether or not he would enjoy a two year term. It is not reasonable to expect any president to undertake any work for the Association unless he can feel reasonably sure that he will have time to complete it before he goes out of office. In general, a man elected for two years will realize his opportunity and take his duties more seriously than one elected for a short term with no particular certainty of re-election."

Negative. "The By-Laws of the Association provide that the president and vice-presidents "shall be eligible for re-election but not for more than two consecutive terms." In the early years of the Association it seemed wise to the

respective presidents to decline in advance the nomination for a second term for the following reasons: (1) in order that, from the very outset, there might be no appearance of undue centralization of power; (2) in order that, as rapidly as possible, there might be a geographic distribution of responsibility and corresponding development of interest in the work of the Association all over the country; (3) in order that the presidential honor might be conferred upon as many as possible of the large number of highly worthy candidates. It seems perfectly evident to me that these objects have been attained to a higher degree by the administrations of the Association's ten presidents than would have been the case if there had been only half as many presidents and each had served two years. Even the interest which has been maintained through these really democratic elections each year should not be lost sight of in this connection.

However, there is weight to the contention that a president who has an aggressive program may find a single year too short a time in which to work out his plans, and it may well be that the precedent followed in the first decade has become too rigid and is in need of modification. This can easily be done by simply giving wide publicity to the fact that the by-laws now provide for reelection and that the growing complexity of the Association's activities may sometimes justify a second presidential term. But this question should be determined in democratic fashion on the merits of the case at any given primary election.

The above plan preserves a desirable flexibility, whereas the proposal to amend the by-laws so as to make the presidential term two years would establish a rigidity from which there would be no escape. For example, the time will soon come when the growth of the Association through its sections (now sixteen in number) will warrant the selection of a president from the far south or the far west, but it would probably be unwise for a long period yet to have the chief executive officer for two consecutive years so far from the center of Association activities. The same considerations apply to our Canadian membership. We would do honor to ourselves as well as to our neighbors on the north if we should sometime select a president from among them, but neither would they nor we think it wise to take such action for a two-year period. The success of the Association has been due in no small measure to the flexibility and democratic adaptibility of its machinery and many of us would regret exceedingly to see these characteristics restricted by the proposed amendment."

This notice of the proposed amendment with two points of view is submitted to the members of the Association, in accordance with Article VIII, Section 1, of the By-Laws.

W. D. CAIRNS, *Secretary-Treasurer.*

THE MAY MEETING OF THE MINNESOTA SECTION.

The regular spring meeting of the Minnesota Section was held at St. John's University, Collegeville, Minnesota, on Saturday, May 16, 1925. Professor W. H. Kirchner, University of Minnesota, the Chairman of the Section, presided.

The attendance was 45 at the dinner, while 73 attended the reading of the papers, and included the following 21 members of the Association: W. O. Beal; A. Bogard; Sister Brigetta; W. H. Bussey; Elizabeth Carlson; H. H. Dalaker; Gladys Gibbens; W. L. Hart; D. Jackson; W. H. Kirchner; W. H. McEwen; E. L. Mickelson; M. A. Norgaard; G. C. Priester; H. L. Rietz; R. R. Shumway; H. L. Smith; F. J. Taylor; Ella Thorp; A. L. Underhill; G. Winkelmann.

The following officers were elected for the coming year: Chairman, M. A. NORGAARD, St. Olaf College, Northfield; Secretary, A. L. UNDERHILL, University of Minnesota.

A motion was passed, expressing the appreciation of the Section for the hospitality of St. John's University.

The 1926 meeting will be held in May at St. Olaf College, Northfield.

The following six papers were read:

(1) "A survey of recent progress on random sampling," by Professor H. L. RIETZ, University of Iowa, National President of the Association, 1923-24.

(2) "The dynamics of correlation," by Professor DUNHAM JACKSON, University of Minnesota.

(3) "A survey of mathematics in the Benedictine Order," by Rev. GILBERT WINKELMANN, O. S. B., St. John's University.

(4) "A paradox in the realm of college grades and honor points," by Professor W. H. BUSSEY, University of Minnesota.

(5) "An experiment with prognostic test in college freshmen mathematics," by Professor M. A. NORGAARD, St. Olaf College.

(6) "Anti-polars in perspective," by Professor W. H. KIRCHNER, University of Minnesota.

Abstracts of papers, numbered as in the above list of titles, follow:

1. The plan of this paper was to give first a general view of those parts of the theory of random sampling which constitute an appropriate setting for remarks on recent progress along this line. The paper first dealt with the Bernoulli theory as the simplest sampling theory and next considered Bernoulli, Poisson and Lexis distributions. With this setting the nature of the extension of the theory by Coolidge published in 1921 was discussed. The paper next dealt with the Tchebycheff inequality and discussed the generalization of this inequality published by Pearson in 1919 and the further generalizations by Camp and Midell published in 1922. The paper next dealt with the inversion of

the Bernoulli-Laplace theorem and with the recent contributions to this subject. The final section of the paper was devoted to the recent contributions of the Pearson school of statisticians to the problem of determining the distribution of averages obtained from small samples.

2. This paper was an elementary account of the principal axes of a statistical distribution, and of the geometric interpretation of coefficients of correlation as based on the formulas pertaining to ellipsoids of inertia in dynamics. The treatment was general, being in no way restricted to normal distributions.

3. The monastery of Monte Cassino was founded by St. Benedict in 529. Professor Winkelmann pointed out that in fourteen centuries over three hundred Benedictines are recognized as having made contributions to the advance of mathematics. Eminent among these were: the venerable Bede for his work on the calendar and on finger calculation; Adelard of Bath, one of the greatest of the translators from Arabic into Latin; Alcuin, because his school and those of his followers led directly to the founding of the University of Paris; and Gerbert of Aurillac (Pope Sylvester II) the most learned man of his time, and, because of his treatises on arithmetic and geometry, perhaps the greatest mathematician of the Order. Reference was made to the Benedictine University of Salzburg, to a long line of skillful teachers, and to mathematicians who, like Galileo and Fourier, came under Benedictine influences although less intimately connected with the Order.

4. In many colleges, requirements for graduation are expressed quantitatively in terms of credits and qualitatively in terms of grades and honor points. In the marking system referred to in Mr. Bussey's paper there are four so called passing grades, A, B, C, D, and a grade F to represent failure; and honor points are given to students according to the following schedule:

With every credit of grade A the number of honor points given is 3,
With every credit of grade B the number of honor points given is 2,
With every credit of grade C the number of honor points given is 1,
With every credit of grade D the number of honor points given is 0,
With every credit of grade F the number of honor points given is -1.

The paper was an attempt to answer questions similar to the following: "If every freshman in a college takes three courses, and if every teacher of those freshmen gives out grades A, B, C, D, F approximately according to a specified grade distribution schedule (for example, 5 per cent of grade A, 15 per cent of grade B, 45 per cent of grade C, 20 per cent of grade D, and 15 per cent of grade F), how many of the freshmen may reasonably be expected to have an average grade of C, that is an average of one honor point per credit, for a year's work."

5. In colleges where sectionizing of freshman classes has been attempted, the grouping has generally been done at the end of the first semester or at the end

of the first six weeks. This delay seems very undesirable; and especially is it detrimental to the bright freshman, who through lack of proper competition may develop an improper attitude to study. For three years Professor Nordgaard has tried out the reliability of a prognostic test, covering the fields of arithmetic, algebra and geometry and given at registration time. He has worked out the correlation between scores thus obtained and the first semester's work in freshman mathematics. Generally the Pearson correlation coefficient falls between .40 and .50, which is somewhat higher than that shown between semester results and tests on current work given at the end of five weeks. Two years the students were sectionized according to the results of the prognostic tests, and one year they were left unsectionized, in order to examine the problem from a different angle.

A. L. UNDERHILL, *Secretary.*

THE COURSE IN STATISTICS IN THE MATHEMATICS DEPARTMENT[1]

BY A. R. CRATHORNE, University of Illinois

In these days of crowded classrooms and diversified curricula the duplication or overlapping of courses has become an important factor in collegiate economics. Many an administrative officer in reviewing the courses in the catalog has paused to consider the various courses containing some form of the word statistics—economics statistics, educational statistics, mathematical statistics, vital statistics, theory of statistics, statistical biology, statistical mechanics, or just "statistics"—and has exhibited undue curiosity as to the amount of overlapping and the possibility of combining or in some way reducing the number of courses. In some institutions this has led into a discussion of the advisability of the formation of a department of statistics.

A great deal of the blame for this repetition of the word statistics in the catalogs can be placed upon the word "statistics" itself. It is not what in mathematics we call "well defined." To many people the first syllable is stressed, the relation with the word "state" is uppermost so that a sociological or economic meaning is given to the word. To others the word has a broader meaning in that it is thought to be concerned with the study of methods for finding the salient characteristics of quantitative data without making any statement about the subject matter of the data. In most dictionaries one finds two or more definitions which usually include these two. It is not my purpose to discuss the precise definition of statistics, or to differentiate between statistics, statistical methods, or theory of statistics, but simply to discuss some features of the courses in statistics given in the mathematics department.

[1] Read Dec. 31, 1925, at the Kansas City meeting of the Mathematical Association of America.

I know of no better introduction to this part of the paper than to quote the carefully worded statement by the committee on the mathematical analysis of statistics of the National Research Council in the preface to its *Handbook of Mathematical Statistics*;—"The study of a problem by statistical methods usually involves three stages: (1) the collection of material or data; (2) the mathematical analysis of the data thus collected; (3) the interpretation of the results, for the particular purpose in view. As to stage (1), the best methods of collecting data depend almost entirely on the nature of the particular field of inquiry. . . . . The same is true in regard to stage (3): the problems connected with the interpretation of statistical results are necessarily very different in different fields of inquiry. . . . . The problems of stage (2), on the other hand, are in a sense common to all fields of statistical inquiry. Whatever the content of the data may be, the form of the mathematical analysis is essentially the same."

The courses in statistics given in the mathematics department should be confined in most part to the second stage although there are times when the mathematical statistician is consulted as to methods of collection of data and more often as to interpretation of results. College courses emphasizing these stages should be given by the departments of education, economics, or sociology and should be considered as courses in education, economics and sociology. Biologists and anthropologists have long recognized the principle implied here and have invented the words biometry and anthropometry. We might ask the economists to introduce the word econometry, but it would be very difficult to argue them out of their rather well grounded historical claim to some share in the word statistics.

The courses in mathematical statistics vary almost as much as courses in mathematics. There are courses for freshmen, sophomores, juniors and seniors and purely graduate courses, but in general they can be grouped under two heads—elementary courses and advanced courses, using the prerequisite of calculus to mark the rather vague boundary line between the two. It is not my purpose to discuss in any detail the content of these courses. In a paper[1] before this association some three years ago, Professor Rietz considered this phase of the statistics course quite fully. In general the content of the elementary course can be broadly sketched by the mention of a few topics: averages, measures of dispersion, frequency distributions, normal and binomial distributions, interpolation, sampling, correlation. By adding to this list the names of Bayes, Lexis, Tchebycheff, Pearson, Edgeworth, Yule, Charlier, Sheppard, Bortkiewitsch, R. A. Fisher, and the topics associated with these names, we sketch in broad outline the advanced course. This does not mean that the

[1] On the Subject Matter of a Course in Mathematical Statistics, by H. L. Rietz, (*1923*, 155-166).

two courses overlap to any extent. For example, the topic "frequency distributions" should occur in any course in statistics, elementary, intermediate or advanced, but in the advanced course it takes on a meaning that would be entirely lost to a student of the other courses.

As for a standard course, even the elementary course has not become standardized like the courses in calculus for example. Very few of those who teach the subject give the same course two years in succession. The students who take the first college course in mathematical statistics are usually those who do not profit so much from lectures as they do from a text book. The subject of text books in statistics is a timely one. In the last year or two there has been almost a deluge of texts on statistics, and the theory of probability. Many of these are excellent but the best are written for students in other departments—economics, education, psychology, sociology. Teachers of statistics from the mathematical side seem to be too individualistic to unite to any extent on a text book. Whenever two of them get together, the conversation usually turns to the book that was used last year, the book now in use and the book that will be tried out next year. The great American text in mathematical statistics still remains to be written; at least it is not yet published. Sometimes the same text is used for a course in the mathematics department and a course in some other department, usually economics. One might look for overlapping here, but with experienced instructors two very distinct courses would be given—one in mathematics, the other in economics. One of the good English books for a junior-senior course in mathematical statistics is really a text book in psychology.

As for the students who take the first course in statistics, they are of several classes. Some few elect it as a general culture course. Many students in the mathematical group elect it to fill out the requirement for a major. Then there are the students in other groups who use the subject as a sort of technical elective. There is one large group of students which seems to be peculiar to the subject. They are older men, often graduate students in some other field with little or no mathematics, who have discovered that they have come to a point where they are very much handicapped by their lack of mathematical training. It is hard to tell such men to review their high school mathematics and then take two years of college mathematics. It is equally hard to resist the request of their department or college for a combined course in mathematics and statistics suitable for this type of student. The mathematics instructor, right down in his heart, takes a somewhat Machiavellian satisfaction out of the plight of these students, but he has enough missionary spirit to compromise and the result is a course with prerequisites: "college algebra and junior standing, or upon consultation with the instructor." There is another and worthier kind of satisfaction for the mathematics instructor in

connection with this statistics course. It comes from those students in the course who happen to have had a course in calculus. The way in which they stand out among the others even when no calculus is used in the course is very gratifying. This seems to be true in other courses in statistics besides those given in the department of mathematics. At the University of Illinois we give each year a course requiring college algebra and junior standing, but year by year the number of students who have had calculus has increased until it has reached a point where there should be two kinds of first courses in statistics—one for those who have had calculus and one for those who have not had it.

Now and then we have the serious and earnest student who takes the elementary course in statistics on top of the minimum amount of mathematics. He often develops into a man who acquires a skill in finding averages, measures of dispersion, coefficients of correlation and other constants of statistics. Substitution in a formula is to him the height of mathematics and the conditions under which the formula was derived are of no particular moment. He calculates columns of probable errors without a thought as to the kind of distribution with which he is working. And correlation coefficients, how he can calculate and interpret them! Here, for example, is a correlation of 0.7 between a time series of prices of pig iron and one of prices of pork. That this correlation has a tremendous and deep lying significance is easily shown. Just look at the probable error, which is .04463!

From many points of view, the most interesting student in the statistics courses is the student who is preparing to become a statistician. Too often he decided on this profession very late in his college course and thinks that a single course in statistics will be sufficient preparation. If he selects the course in the mathematics department, he is usually a student in the mathematics group and has seldom had much economics. Such a man is very one-sided in his preparation to do statistical work. He has an exaggerated idea of the usefulness of mathematics in all departments of statistics and comes late to an appreciation of the fact that knowledge of the subject matter of the statistical problem is very necessary. Then he is often disappointed when he starts work and finds that at first he doesn't seem to have much use for his advanced mathematics. There is a whisper going about the campus nowadays that statistics is a good thing to go into and sometimes it is necessary to dampen the enthusiasm of an undergraduate who expects immediately upon graduation to integrate, interpolate, and correlate his way to a five-figure salary. Statistics is a good thing for the right person to look forward to, but, like any other worthwhile profession, it needs careful preparation and a hard climb afterwards. The prospective statistician should keep in mind that there are three phases of the general problem of statistics and that the course in the mathematics de-

partment is mainly concerned with only one. If he could arrange his college course from the beginning, he should elect mathematics and accounting in his freshman year, mathematics and economics in his sophomore year. In his junior year he should take some mathematical statistics and mathematics of finance and courses in economics. In his senior year he should take a course in economic statistics and somewhere in his course, even if he has no particular liking for natural science, he might take some courses in biology to get the point of view of the biological statisticians or biometricians as they call themselves. This work of course takes up only a small part of his time. It is assumed that he is taking a general college course with opportunity to elect the courses mentioned. This sketchy suggestion as to the arrangement of a college course to suit the needs of a future business statistician does not have in mind the research worker who will need a more intense and pointed preparation in the graduate school.

In the advanced course in mathematical statistics there is not much question of overlapping. It is a course in pure and applied mathematics. The students are usually graduate students and are either from the mathematics department or are taking mathematical statistics as a minor. Now and then a student appears who is looking forward to a research position in statistics. I have indicated roughly a few of the characteristics of this course but I doubt if any of the men who give it ever follow the same outline two years in succession. The theory of statistics as we now know it is rather young and undeveloped. The growing pains are very evident. We notice their effect in the symbolism and language of statistics which have been adopted in considerable part direct from the theory of probabilities after hard usage in the theory of errors.

In his review of Keynes' *Theory of Probability*, Sheppard, the well-known English statistician, makes some pertinent remarks bearing on the relation between the theory of probability and statistics.

"The theory of rational belief is one thing: the mathematical theory of probability—which is practically dead—except as a mathematical exercise . . . . its place having been taken by the frequency calculus—is another: and the science of statistics—an inductive science, based or to be based on observation and experiment, and using the frequency calculus as its deduction machine—is another. This science is being built up slowly and Mr. Keynes does not give much help by putting up fragments of the old theory of probability and knocking them down again. . . . . It is quite true that statisticians do use such phrases as "probable error" which seem to connote probability. But probability, if the word is used in the ordinary sense, is one thing, frequency is another. You cannot infer frequency from probability, nor probability from frequency, unless in either case, probability means frequency, which is practically what it does mean to the statistician."

A great deal of what Sheppard has said may be summed up by saying that for many years in its youth the theory of statistics was dressed by its parents in old clothes belonging to a kind uncle called the theory of probability. The nephew endured it patiently for many years, but of late having earned a little

money of his own, he is busy making inquiries as to the best tailors and haberdashers and hopes soon to surprise his uncle who has retired to his country home on a pension. To put it more concisely, we can say that one of the important problems of the present and future is the untangling of the theories of probability and statistics.

A fundamental problem in statistics which is badly in need of this untangling treatment is as follows: An event has happened p times and failed q times out of $p+q$ trials. We know nothing of the underlying frequency of the event. We wish to know what will happen in the next m trials. This problem leads us to the subject of inverse probabilities and the not-to-be-suppressed controversy over Bayes' theorem. I cannot do better on this point than quote R. A. Fisher:[1]

"Several reasons have contributed to the prolonged neglect into which the study of statistics in its theoretical aspects has fallen. In spite of the immense amount of fruitful labor which has been expended in the practical applications, the basic principles of this organ of science are still in a state of obscurity and it cannot be denied that during the recent rapid development of practical methods, fundamental problems have been ignored and fundamental paradoxes left unsolved. This anomalous state of statistical science is strikingly exemplified by a recent paper in which one of the most eminent of modern statisticians presents what purports to be a general proof of Bayes' postulate, a proof which in the opinion of a second statistician of equal eminence "seems to rest upon a very peculiar—not to say hardly supposable—relation." . . . . It has happened that in statistics a purely verbal confusion has hindered the distinct formulation of statistical problems: for it is customary to apply the same name, mean, standard deviation, correlation coefficient, etc., both to the true value which we should like to know, but can only estimate, and to the particular value, at which we happen to arrive by our methods of estimation. It is this last confusion in the writer's opinion, more than any other, which has led to the survival to the present day of the fundamental paradox of inverse probability, which like an impenetrable jungle arrests progress towards precision of statistical concepts. The criticisms of Boole, Venn and Chrystal have done something towards banishing the method at least from the elementary text-books of Algebra; but though we may wholly agree with Chrystal that inverse probability is a mistake (perhaps the only mistake to which the mathematical world has so deeply committed itself), there yet remains the feeling that such a mistake would not have captivated the minds of Laplace and Poisson if there had been nothing in it but error."

Fisher is constructive in his criticism, for the quotations are taken from the beginning of his memoir in which he reviews with care Bayes' theorem and the recent criticism, and proposes his method of maximum likelihood.

Another illustration showing how the science of statistics is still in the process of being slowly built up, is given by the theory of skew frequency curves and surfaces. Every student of mathematical statistics knows of the attempts to represent frequency distributions analytically and of the progress during the last thirty years which has produced the two systems of curves usually associated with the names of Pearson and Charlier. The corresponding problem of representing skew frequency surfaces[2] analytically has had a slower growth. It would seem natural that Pearson after writing his first paper on univariate

[1] On the Mathematical Foundations of Theoretical Statistics, by R. A. Fisher, *Phil. Trans.*, vol. 222 A, p. 309.

[2] Skew Frequency Surfaces, by Karl Pearson, *Biometrika*, vol. 15, p. 223.

skew variation should turn to the question of bivariate variation. In the case of normal distributions one can multiply together two normal functions in different variables and by rotation of the axes arrive at the normal surface and with it our present day theory of correlation. In Pearson's first attempts to solve the problem of bivariate skew variation the results were in a way negative. By analysing skew correlation data he surmised from the study of special cases that we could not proceed by multiplying together two skew functions and following with a rotation of axes. Later by using a double hypergeometric series he tried to determine a surface which was related to the double series in somewhat the same way that his skew curves were related to the single hypergeometric series. From the resulting differential equations he inferred that the regression lines were not straight lines but were cubic curves intersecting at a point which was not the mean of the surface. These differential equations were of the form

$$\frac{1}{z}\frac{dz}{dx} = \frac{\text{cubic in } x \text{ and } y}{\text{quartic in } x \text{ and } y}, \qquad \frac{1}{z}\frac{dz}{dy} = \frac{\text{cubic in } x \text{ and } y}{\text{same quartic in } x \text{ and } y}.$$

During the next twenty years various men worked on the problem with Pearson, —Filon, Isserlis, Rhodes[1]—with results that were special and no very satisfactory advances toward the solution of the general problem were made. Gradually Pearson drifted away from his attack on the skew correlation problem by means of his differential equations to one in which he assumed the forms of the regression and scedastic curves, and from this starting point, under reasonable hypotheses, he hoped to arrive at the equation of the corresponding frequency surface. This method of attack was suggested to Professor Narumi, a Japanese mathematician, and the result was the interesting series of memoirs in *Biometrika*, as yet uncompleted, in which Narumi[2] has made great advances in the problem of bivariate skew frequency distributions.

A recent advance of far reaching importance in practical statistics is the publication of tables of the incomplete gamma function[3] after some twenty years of preparation. These tables give the probability of the occurrence of a deviation from the mean that is greater than any given deviation. They do for certain skew frequency curves what the ordinary normal probability table does for the Gaussian curve. The tables do not cover all Pearson's skew variation curves, but they cover a very large class of some hundreds of curves of which the normal curve is a single member. Tables of incomplete beta functions are desirable. The gamma function tables are two-fold while the

[1] On a certain skew correlation surface, by E. C. Rhodes, *Biometrika*, vol. 14, p. 355.

[2] On the general forms of bivariate frequency distributions which are mathematically possible when regression and variation are subjected to limiting conditions, by Seimatsu Narumi, *Biometrika*, vol. 15, p. 77, and p. 209.

[3] Tables of the incomplete gamma function, edited by Karl Pearson, London, 1922.

beta function table would have to be three-fold so the cost of computation and printing will be well nigh prohibitive. Right here is a very practical problem for present and future workers in the mathematical side of statistics. We all know how the beta function can be expressed in terms of gamma functions, but what about the incomplete beta and gamma functions? Can the incomplete beta function be expressed exactly or approximately in terms of the incomplete gamma function?[1]

Progress along purely mathematical lines is gratifying in the highest degree to workers in theoretical statistics who feel that they have a worthwhile field of their own and rather resent being considered as hewers of wood and drawers of water for others. Nevertheless they all take added pleasure in those advances in their science which prove useful in a practical sense, though one might add that any new advance in the theory of statistics will be sure sooner or later to have its application in some other field. This is well illustrated in the work now being done in the statistical department of the Bell Telephone laboratories. For an example, we roughly sketch some of the applications of statistics to the carbon microphone[2] which is the essential part of the telephone transmitter. Something over one and a half million transmitters are manufactured every year by the Bell system. Variations in the efficiency of the carbon microphone have many causes. There are all the variations connected with the carbon particles, the variations due to the process of assembling the parts, and to differences in the individual parts, variations due to temperature, humidity, personality, and last but not least the variations in the human ear which measures the efficiency. The physicist usually works in a laboratory under controlled conditions, but the telephone engineer must work under commercial conditions. It is not possible to inspect every transmitter. Instead, random samples are used. Certain variations will be found. Can these be explained by the laws of chance? If not, what is the explanation? Is there a general trend upward or downward in any particular characteristic? If there is, what is the cause? Then comes the important economical question: "What should be the size of the sample to be inspected?" To answer these questions satisfactorily is the work of the statistics department. In most problems the first step is the determination of the law of distribution about some mean value of the variable being investigated, and from this law to find the most probable value and the frequency of values occurring between two given limits and then to answer the question, "Are the data such that the variation would follow from a random system of causes?" Here it is necessary to call in the aid of the whole subject of frequency curves skew and non-skew, the theory of random sampling

[1] The Fundamental Problem of Practical Statistics, by Karl Pearson, *Biometrika*, vol. 13, p. 16.

[2] Some Applications of Statistical Methods, by W. A. Shewhart, *Bell System Technical Journal*, vol. 3, No. 1.

for large and small samples. In fact one of the men in the Bell Telephone laboratories wrote me that he did not recall many theorems in mathematical statistics which had not been used at some time or other in their study of the carbon microphone.

In a paper in the current number of the *Journal of the American Statistical Association* Dr. Shewhart[1] has an interesting article on the use of statistics in maintaining the quality of a manufactured product. When the production of a single kind of article runs into the millions it is a great advantage to be able to discover the existence of causes of non-random variation in a product as early as possible. Among other things in this paper, the author shows that the Pearson goodness-of-fit test is a very useful tool in this problem.

The size of the sample is a question of great economic importance. At the present time a theory of small samples is in process of development which promises to be of the greatest value.[2] Then the problem of making the most efficient use of data is an important economic problem. The cost of analysis of data is small as compared with the collection of the data. Fisher[3] gives a simple illustration of economy in the use of data. In estimating the standard deviation of a normally distributed population from an ungrouped sample, we may use one of two methods—the mean error method or the root-mean-square error method. The use of the latter method with a sample of 100 is equivalent to the use of the former with a sample of 114. This is not a negligible economy when the total number of observations runs into millions.

These references to recent advances in the purely theoretical side on the one hand and to the extremely practical on the other are but what might be the beginning of a very long list. They are simply those things that have especially interested me during the past year or so, and are touched upon here to give emphasis to the fact that the course in statistics in the mathematics department should be considered as a course in mathematics in its best sense —not a mathematics of drudgery and long computations,—a course which leads to a frontier where either the pure mathematician or the practical statistician may find a fertile field. The teacher of even the most elementary course in statistics in the mathematics department is most interested in the students who show aptitude on the real mathematical side. He does his best work when teaching mathematics. There is in the mind of some students and some other people the idea that a course in statistics is a course in statistics. Some people expect the instructor in the mathematics of statistics to teach economics,

[1] The Application of Statistics as an Aid in Maintaining Quality of a Manufactured Product, by W. A. Shewhart, *Journ. Amer. Stat. Assn.*, vol. 20, p. 546.

[2] The Probable Error of a Mean, by Student, *Biometrika*, vol. 6, p. 1. On the "Probable Error of a Coefficient of Correlation Deduced from a Small Sample," by R. A. Fisher, *Metron*, vol. 1, part 4, p. 82.

[3] *Phil. Trans. Roy. Soc.*, vol. 222, series A, p. 316.

biology or what not. Some others accuse him of teaching economics, biology or what not and teaching it very badly.

The serious part of this whole matter in my mind is not a question of overlapping of courses but a question of coördination among departments. In a physical and geographical sense there can be considerable coöperation. It is not a mere coincidence that the increase of interest in statistics has come at the same time as a decrease of drudgery due to improvements in statistical apparatus. A statistical laboratory is a necessity in the teaching of any course in statistics. Expensive computing machines and readily accessible library equipment may be collected at a place convenient to all departments where it can be used in common. There will be real economy here. Perhaps the best way in which the mathematics department can coöperate is to teach mathematics in our courses in statistics and at the same time to remember and point out that we are giving a narrow and one sided view of the broad field of statistics which should be supplemented by other courses. If the mathematician thinks it too presumptuous to suggest that a course in mathematical statistics be made a prerequisite for other courses in statistics, he may ease his conscience by reminding himself that if the mathematics instructor does not stress the mathematical side no one else will.

THE STURM AND FOURIER–BUDAN THEOREMS AND MIXED DIFFERENTIAL–DIFFERENCE EQUATIONS

By TOMLINSON FORT, Hunter College

I recently tried with indifferent success to teach the Sturm and Fourier-Budan theorems to undergraduates from the proofs given by Dickson in his *First Course in the Theory of Equations*. I then substituted the proofs that are given here. It is to be noticed that they are primarily graphical and with the aid of the black board are easily explained. I have now used them in two different classes with success. In fact the students seem a little amused at the idea that previous classes had found the theorems hard. These proofs are given to teachers of the theory of equations for what they are worth. It is also to be noticed that in section 4 wide generalizations of the Fourier-Budan theorem are given. These are simple but so far as I know are new and may be found of mathematical interest. To one who has read the previous sections proofs would be very easy and so I have not given them in detail.

1. Sturm's Theorem as given by Dickson[1] is:

"Let $f(x)=0$ be an equation with real coefficients and without multiple roots. Modify the usual process of seeking the greatest common divisor of $f(x)$ and

[1] *First Course in Theory of Equations* p. 76.

its first derivative $f_1(x)$ by exhibiting each remainder as the negative of a polynomial f_i:

$$f=q_1f_1-f_2\,, \qquad f_1=q_2f_2-f_3\,, \qquad f_2=q_3f_3-f_4\,, \cdots\,, \tag{1}$$

$f_{n-2}=q_{n-1}f_{n-1}-f_n$ where f_n is a constant $\neq 0$. If a and b are real numbers $a<b$, neither a root of $f(x)=0$, the number of real roots of $f(x)=0$ between a and b is equal to the excess of the number of variations of sign of

$$f(x)\,, \qquad f_1(x)\,, \cdots\,, f_{n-1}(x)\,, \qquad f_n(x) \tag{2}$$

for $x=a$ over the number of variations of sign for $x=b$. Terms which vanish are to be dropped out before counting variations of sign."

Look upon the Sturmian functions, $f_0(x)=f(x), f_1(x), \cdots, f_i(x), \cdots, f_n(x)=$ constant, as functions of the subscript i with x as a parameter. Graph this function of i by simply erecting at the points, i, on a horizontal axis vertical straight line segments of length f_i. The length of each of these line segments is of course a function of x. Connect their ends by straight lines getting a figure somewhat as follows:

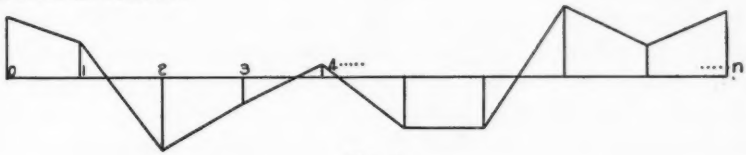


FIG. 1

The broken line gives a picture of the function f_i. The number of variations in sign is the number of points where the broken line crosses the axis. We call them nodes.

Now this broken line graph never touches the axis between 0 and n without crossing it. For if $f_i=0, f_{i-1}=-f_{i+1}=0$ and all three are not zero at the same time since for that value of x, f_0 and f_1 would vanish, but there are no multiple roots. Hence when x varies the number of nodes does not change by having two, as it were, lift out or enter thus (Fig. 2). All that can happen is for the nodes to move on the axis. The only way for their number to change is through entrance or exit of nodes at the point, 0, f_n being a constant. Let x increase from a to b. Every time x passes through a root, x_0, of $f(x)$ a node disappears from the interval $(0,\ n)$. For remember that $f_0=f(x)$ is a polynomial and that $f_1\equiv f'(x)$. In the neighborhood of x_0 for values of x less than $x_0, f(x)$ and $f'(x)$ are of opposite sign and for values of x greater than x_0 of the same sign. Our figure must change from (Fig. 3) to (Fig. 4). A node has disappeared. Hence as x increases from a to b the decrease in the number of nodes (*varia-*

FIG. 2

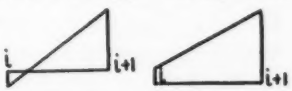


FIG. 3

FIG. 4

tions in sign) is the same as the number of roots of $f(x)$ lying between a and b and the theorem is proved.

2. There seems to me no need of a distinct treatment where there are multiple roots. In this case consider $f(x)$ and $f'(x)$ as having their highest common factor removed, then as not having it removed. The Sturmian functions in the second case differ from those in the first only by being multiplied by this factor. This will not change the number of variations in sign at a or b. The theorem in the case of multiple roots follows.

3. The Fourier-Budan theorem as stated by Dickson[1] is:

"Let a and b be real numbers $a<b$ neither a root of $f(x)=0$ an equation of degree n with real coefficients. Let[2] V_a denote the number of variations of sign of $f(x)$, $f'(x)$, $f''(x)$, $\cdots$, $f^n(x)$ for $x=a$ after vanishing terms have been deleted. Then V_a-V_b is either the number of real roots of $f(x)=0$ between a and b or exceeds the number of those roots by a positive even integer. A root of multiplicity m is here counted as m roots."

A proof of this theorem requires only slight modifications of that given for Sturm's theorem. The proof will be sketched only. Here the functions are $f_0(x)\equiv f(x)$, $f_1(x)\equiv f'(x)$, $\cdots$, $f_n(x)\equiv f^n(x)\equiv$constant. Construct a figure in all ways analogous to that for the Sturmian functions. Two things, however, are different. First; an even number of nodes may pass out when x increases from c to d although no root of $f(x)$ lies between c and d, crossing points of the kind (Fig. 5) lifting out as it were, the figure changing from 5 to (Fig. 6). The

FIG. 5

FIG. 6

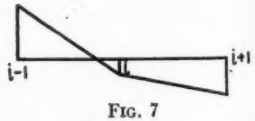


FIG. 7

reverse, however, never happens as a figure changing from the type 6 to 5 or from 6 to (Fig. 7). This would require $f_i(x)$ and $f_i'(x)$ both to be of the same sign immediately preceding a root of $f_i(x)$. The second difference between the

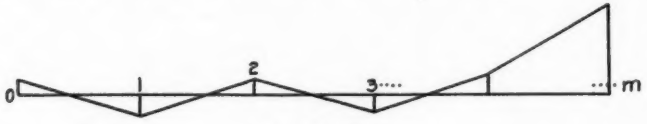


FIG. 8

Fourier-Budan and Sturm theorems is that in case $f(x)$ has a multiple root of order m, $f_0(x)$, $\cdots$, $f_{m-1}(x)$ all vanish at the same time. The figure changes

[1] *First Course* . . . . , p. 83.

[2] Probably V_x . . . . for $x=x$ was intended thus explaining both V_a and V_b.

from (Fig. 8) to (Fig. 9). The reasoning is identical with that already employed as each function is the derivative of the preceding. A number of nodes equal to the multiplicity of the root have disappeared and the theorem is completely proved.

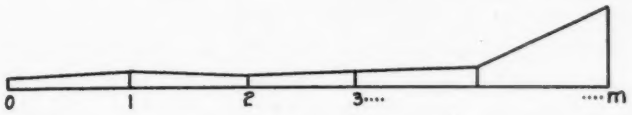


FIG. 9

4. A consideration of the proof of the Fourier-Budan theorem shows that it depends essentially on the fact that y_{i+1} has the sign of y_i' in the neighborhood of a root of y_i. We immediately generalize by letting

$$y_{i+1}=p_i(x)y_i'+q_i(x)y_i\ . \tag{3}$$

We no longer require that y_0 be a polynomial but only that it be analytic[1] $a\leqq x\leqq b$ and of p_i and q_i only that they be analytic and that p_i be positive over the interval. We suppose that we proceed to calculate first y_1 then y_2 etc. and that we reach a y_n which does not vanish so long as $a\leqq x\leqq b$ and that a and b are such that $y_0(a)\neq 0$, $y_0(b)\neq 0$. We now have what we shall call a Budan sequence and from it we determine V_a-V_b an integer equal to the number of roots of y_0 on the interval in question or exceeding it by an even positive integer just as for a polynomial by means of its derivatives.

Equation (3) is a mixed differential-difference equation of the first order in differences and first order in derivatives.

We next generalize by increasing the order of (3) thought of as a difference equation. We write

$$y_{i+1}=p_i(x)y_i'+q_i(x)y_i-P_{i-1}(x)y_{i-1}\ , \tag{4}$$

where the p_i and q_i are as before and $P_i\geqq 0$ is analytic and $P_{i-1}(x)\equiv 0$.

Suppose that y_i is calculated by successive steps from an initial and analytic y_0 and that y_n retains a fixed sign as x varies from a to b. Here the functions $y_0, y_1, \cdots, y_n$ are readily shown to form a Budan sequence. It is only necessary to observe that if two nodes should come in from the vanishing of y_i that y_{i-1} and y_{i+1} would be of the same sign just before this root of y_i is reached and that this would be the sign of y_i'. This is the already familiar impossibility. The situation at y_0 is as formerly.

Now turn equation (3) around and write

$$p_{i+1}(x)y_{i+1}'+q_{i+1}(x)y_{i+1}=y_i\ , \tag{5}$$

[1] The requirement that y_0, p_i and q_i be analytic is more than is necessary but it is put down as sufficient and general enough for interesting cases.

where the coefficients are as formerly. We note particularly that $p_i(x)>0$. y_0 is chosen at pleasure so long as it is integrable and retains a fixed sign as x varies from a to b. Equations (5) can be solved for the functions y_i successively, choosing each time any particular solution. We have a set exactly like that obtained from (3) only in reverse order. In other words we have a function y_i such that the change in the number of nodes of y_i on the interval 0 to n when x increases from a to b is the number of roots of $y_n(x)$ on the interval a to b or exceeds that number by an even integer, neither a nor b being a root of $y_n(x)$.

We next write

$$p_{i+1}(x)y''_{i+1}+q_{i+1}(x)y'_{i+1}+r_{i+1}y_{i+1}=y'_i+P_i(x)y_i\ , \tag{6}$$

where $p>0$. If

$$p''_{i+1}+2P_ip'_{i+1}+(P'_i+P_i^2)p_{i+1}-q'_{i+1}-P_iq_{i+1}+r_{i+1}=0 \tag{7}$$

by actual substitution it is seen that (6) is satisfied by a function which in turn satisfies an equation of the type,

$$R_{i+1}(x)y'_{i+1}+S_{i+1}(x)y_{i+1}=y_i\ , \tag{8}$$

where $R_{i+1}\equiv p_{i+1}$ and $S_{i+1}=q_{i+1}-P_ip_{i+1}-p'_{i+1}$. We solve (8) and have a sequence of functions which are a solution of (6) and at the same time a Budan sequence.

We can proceed to equations of higher differential order with a condition on the coefficients similar but naturally somewhat more complicated than the above.

A CROSS-DIVISION PROCESS AND ITS APPLICATION TO THE EXTRACTION OF ROOTS

By DERRICK HENRY LEHMER, University of California

The computer is sometimes confronted with the problem of dividing by a number that is too large to fit in his calculating machine. In this paper we shall present an effective method of overcoming this difficulty, which was used by the writer in calculating the value of the Napierian base to 707 places of decimals, and also give an adaption of the division process for extracting square roots.

There are methods which serve to find the decimal value of a fraction whose denominator is less than twice the key-board capacity of the calculating machine. For a ten-place calculator these methods are of practical use up to perhaps 17-digit denominators. The division method given herewith will be found most useful for denominators beyond this range.

The method is virtually the reverse of a cross-multiplication method given by D. N. Lehmer (*1923*, 67) which may be briefly described as follows. Suppose we are given two large numbers A and B, whose product is required. We break up A and B into equal periods of convenient length. Thus:

$$A = \alpha_1 , \alpha_2 , \alpha_3 , \alpha_4 , \alpha_5 \cdot \cdot \cdot$$
$$B = \beta_1 , \beta_2 , \beta_3 , \beta_4 , \beta_5 \cdot \cdot \cdot ,$$

where the α's and the β's represent the numbers making up the several periods of A and B.

Now let

$$\sigma_1 = \alpha_1\beta_1 \qquad \sigma_2 = \alpha_1\beta_2 + \alpha_2\beta_1 \qquad \sigma_3 = \alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1$$
$$\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$
$$\sigma_n = \alpha_1\beta_n + \alpha_2\beta_{n-1} + \cdot \cdot \cdot + \alpha_{n-1}\beta_2 + \alpha_n\beta_1 .$$

The multiplication then assumes this form:

$$A = \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \cdot \cdot \cdot$$
$$B = \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \cdot \cdot \cdot$$
$$\sigma_1 \quad \sigma_3 \quad \sigma_5 \quad \cdot \cdot \cdot$$
$$\sigma_2 \quad \sigma_4 \quad \sigma_6 \cdot \cdot \cdot$$
$$C = AB = \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \cdot \cdot \cdot$$

These σ's are of course two-period numbers and are written on two lines so as to allow for overlapping . The γ's are formed by adding the periods that appear in the two lines directly above.

For division the process is exactly the reverse; we are given the fraction C/A and we wish to find the quotient B. In other words we are given the α's and the γ's from which we are to deduce the β's.

The general plan of attack is as follows. Since the first period of σ_1 is approximately equal to γ_1 we can determine β_1 from the relation

$$\sigma_1 = \alpha_1\beta_1 .$$

Knowing β_1, the exact value of σ_1 is found by multiplication. Now, since we know the last period of σ_1, we obtain the first period of σ_2 by subtracting from γ_2. Using this value as an approximation to σ_2, we determine β_2 from the relation

$$\sigma_2 = \alpha_1\beta_2 + \alpha_2\beta_1 .$$

Having determined β_2, this same equation enables us to find the exact value of σ_2, the last period of which is used to find, from γ_3, the first period of σ_3 from which β_3 is to be determined from the relation

$$\sigma_3 = \alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1$$

and so on.

For practical use, however, this process must be modified to allow for disturbances arising from two sources: first the uncertainty as to the last digit in any β which is determined from only an approximate value of the corresponding σ, and second, the effect on the first period of any σ caused by the carrying over of a unit or more from the σ immediately following. The modified process is best explained by the use of an example. The reasons for the various steps are easily derived from the cross-multiplication method given above and are not of enough importance to warrant a detailed explanation. It may be remarked that there are several modifications of the above process than can be used successfully. The method given herewith has been found to give the best results.

In this method we make use of a new quantity, σ', which is the result of omitting the first term of the corresponding σ. Thus

$$\sigma_1'=0 \qquad \sigma_2'=\alpha_2\beta_1 \qquad \sigma_3'=\alpha_2\beta_2+\alpha_3\beta_1$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$\sigma_n'=\alpha_2\beta_{n-1}+\alpha_3\beta_{n-2}+\cdots+\alpha_n\beta_1 .$$

Consequently we have $\sigma_n=\sigma_n'+\alpha_1\beta_n$.

The number of digits in each period is chosen as nine for two reasons. First the successive values of the σ''s tend to increase and soon one is reached that is too large to be written in two periods. A unit must therefore be carried over into the preceding period. Now in a ten-place calculator with only twenty places in the carriage, this carrying over of a unit could not be effected if the number of digits in a period were ten. Using nine digits instead, there remain two dials which serve to hold the amount carried over even when the numbers to be divided are as large as 3,000 digits. The second reason has to do with the checking of the values of the successive σ''s in which, as experience shows, errors are most likely to occur. The process of checking is similar to the familiar method of "casting out the nines" used in ordinary multiplication. Instead of using nine for our modulus however we use 1001 which can be cast out rapidly from a nine or eighteen digit number by splitting it up into periods of three digits each and subtracting the sum of the even periods from that of the odd. An added advantage in using 1001 is that an error will not only be detected but frequently located at once.

As an example, let it be required to find the cube root of 83 by dividing

751213767741506747406765351515 3

by

172214946598048117769345011132

which are solutions corresponding to x and y of the Pellian cubic

$$x^3+Dy^3+D^2z^3-3Dxyz=1$$

in which $D=83$. We put down the outline of our work as follows:

	α_1	α_2	α_3	α_4	α_5
$A=$	172214946	598048117	796934501	113200000	000000000
$B=$	β_1	β_2	β_3	β_4	β_5
$C=$	075121376	774150674	740676535	151530000	000000000
	γ_1	γ_2	γ_3	γ_4	γ_5

We first set the number γ_1, γ_2 in the carriage, that is the number

075121376 774150674 .

This we divide by α_1 and we obtain the following quotient and remainder:

$$q_1=436207067 \quad r_1=285927292 .$$

We next multiply α_2 by q_1 and obtain

$$\alpha_2 q_1=260872815 \quad 041442839.$$

We observe that if q_1 were β_1 this two-period number would be σ_2'. Now it can be shown that σ_2' lies between r_1 $\alpha_2 q_1 =$ and $r_1-\alpha_1$. Our r_1 has been properly chosen so as to fulfill this condition. Hence q_1 is β_1 and the above two-period number is σ_2'. We may now put down β_1 and also σ_1 which is $\alpha_1\beta_1$ and which goes directly beneath β_1. The work now appears as follows:

	α_1	α_2	α_3	α_4	α_5
$A=$	172214946	598048117	796934501	113200000	000000000
$B=$	436207067				
	β_1	β_2	β_3	β_4	β_5
	075121376	488223382			
$C=$	075121376	774150674	740676535	151530000	000000000
	γ_1	γ_2	γ_3	γ_4	γ_5

We now set the number r_1, γ_3 in the carriage, that is the number

285927292 740676535

and subtract from it the above value of σ_2'. The remainder when divided by α_1 gives the following quotient and remainder:

$$q_2=145483755 \quad r_2=688031466 .$$

We next calculate $\beta_1\alpha_2+q_2\alpha_2$. By not clearing the carriage after the first multiplication, the second product is added on automatically. In making this second multiplication we take care to set α_2 in the key-board, not q_2. This then gives us

$$\beta_1\alpha_2+q_2\alpha_2=434634747 \quad 004157902.$$

We observe that if q_2 were β_2 this two-period number would be σ_3'. Unfortunately our value of r_2 is too large to fulfill the condition that the above number, which is an approximation to σ_3', shall lie between r_2 and $r_2-\alpha_1$. To remedy this difficulty we observe that if q_2 were changed

from 145483755 to 145483756

then r_2 would be decreased by the amount α_1 and hence its first two digits would be 51. With this value of r_2 the above condition would be satisfied. Making this slight change in q_2 compels us to change the above two-period number

from 434634747 004157902

to $\sigma_3'=434634747 \quad 602206019$

by a turn of the crank in the positive direction.

Having determined in this way that

$$\beta_2=145483756$$

we are now in a position to obtain σ_2. This is done by adding the product $\alpha_1\beta_2$ to the above value of σ_2'. Putting down β_2 and σ_2, the work appears as follows:

α_1	α_2	α_3	α_4	α_5
$A=172214946$	598048117	796934501	113200000	000000000
$B=436207067$	145483756			
β_1	β_2	β_3	β_4	β_5
075121376	488223382			
	285927292	224860015		
$C=075121376$	774150674	740676535	151530000	000000000
γ_1	γ_2	γ_3	γ_4	γ_5

If we add the numbers on the two lines directly above γ_2 we get 774150674 which is γ_2 itself. This then is a check on the work. It should be understood that this is not a check on the σ''s. These numbers must be checked independently as stated above by casting out 1001.

We wish now to determine β_3. The number r_2, γ_4 or

688031466 151530000

is set in the carriage and the value of σ_3' is subtracted from it. The remainder thus obtained is divided by α_1 and the quotient we get contains a unit in the tenth digit which agrees with the supposition that q_2 as we first obtained it was too small by one in the last place. Disregarding this superfluous 1 we try:

$$q_3=471397947 \quad r_3=562208119.$$

We now calculate $\beta_1\alpha_4+\beta_2\alpha_3+q_3\alpha_2$ and obtain

447238319 036881555.

It is seen that we have made a fortunate choice of our r_3 for the above number lies between r_3 and $r_3-\alpha_1$. We therefore have $q_3=\beta_3$ and the above two-period number is σ_4'. Knowing β_3 we may now complete σ_3 by adding the product $\alpha_1\beta_3$ to σ_3'. The work now appears as follows:

α_1	α_2	α_3	α_4	α_5
$A=172214946$	598048117	796934501	113200000	000000000
$B=436207067$	145483756	471397947		
β_1	β_2	β_3	β_4	β_5
075121376	488223382	515816519	589321881	
	285927292	224860015		
$C=075121376$	774150674	740676535	151530000	000000000
γ_1	γ_2	γ_3	γ_4	γ_5

If we add the numbers on the two lines directly above γ_3 we get 740676534 which warns us that there will be a unit carried over from the fourth period.

By carrying out the above process the division of C by A might be extended indefinitely; but if we wish to terminate it with β_5 the last two β's can always be obtained in one step. Proceeding as above, we set the number r_3, γ_5 in the carriage and subtract σ_4'. Dividing by α_1 we try:

$$q_4=667594782 \quad r_4=631106673.$$

We now calculate $\beta_1\alpha_5+\beta_2\alpha_4+\beta_3\alpha_3+q_4\alpha_2$ the first term of which is zero. This gives

791395851 138194941.

We now see that r_4 is too small and hence q_4 should be changed

from 667594782 to $667594781=\beta_4$

if the condition that the above two-period number will lie between r_4 and $r_4-\alpha_1$ is to be fulfilled. By turning the crank one revolution in the negative direction we get

$$\sigma'_5=791395850 \quad 540146824.$$

We next find that $r_4+\alpha_1-\sigma_5'$ when divided by α_1 gives

$$q_5=069249318 \quad r_5=899946348.$$

From this we may assume that $q_6=\beta_6$ with doubt only as to the last digit. Knowing β_4 and β_5 we may complete σ_4 and σ_5. Inserting all these numbers and retaining only the first period of σ_5 the work in its final form appears as follows:

α_1	α_2	α_3	α_4	α_5
$A=172214946$	598048117	796924501	113200000	000000000
$B=436207067$	145483756	471397947	667594781	069249318
β_1	β_2	β_3	β_4	β_5
075121376	488223382	515816519	589321881	803321619
	285927292	224860015	562208118	196678381
$C=075121376$	774150674	740676535	151530000	000000000
γ_1	γ_2	γ_3	γ_4	γ_5

Consequently we have

$$\sqrt[9]{83}=4.36207\ 06714\ 54837\ 56471\ 39794\ 76675\ 94781\ 06924\ 9318.$$

The mistakes in selecting the proper values for q_2 and q_4 might easily have been avoided by forming mentally a rough estimate of σ_3' and σ_5'. For larger divisions however this would not be practical nor indeed necessary since the above procedure solves the difficulty much more readily.

The above process may be adapted for the extraction of the square root of a number. This method depends also upon the proposition that if $n+1$ digits of the square root of a number are known and we wish the root correct to $2n+1$ digits, the remaining n figures may be found by dividing the remainder at that stage of the work by twice the root already found. The general method works equally well whether the number whose square root is to be extracted is an integer or a an irrational number and is a little quicker to perform than division. As in the division process, the method must be modified to allow for carrying over etc. and an explanation is best afforded by the use of an example. The outline of the work may be represented as follows:

$$\sqrt{N}=\beta_1,\ \beta_2,\ \beta_3,\ \beta_4,\ \beta_5\ \cdots$$
$$\cdot\ \sigma_1 \qquad \sigma_3 \qquad \sigma_5\ \cdots$$
$$\sigma_2 \qquad \sigma_4 \qquad \cdots$$
$$N=\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5\ \cdots$$

In this case however the σ's are defined as follows:

$$\sigma_1=\beta_1^2 \qquad \sigma_4=2(\beta_1\beta_4+\beta_2\beta_3)$$
$$\sigma_2=2\beta_1\beta_2 \qquad \sigma_5=2(\beta_1\beta_5+\beta_2\beta_4)+\beta_3^2$$
$$\sigma_3=2\beta_1\beta_3+\beta_2^2 \qquad \sigma_6=2(\beta_1\beta_6+\beta_2\beta_5+\beta_3\beta_4)$$
$$\cdots\cdots\cdots\cdots$$

These values may be obtained from the σ's used in division by changing the α's to β's and simplifying. The σ''s are defined as follows:

$$\sigma_1'=\sigma_2'=0 \qquad \sigma_5'=2\beta_2\beta_4+\beta_3^2$$
$$\sigma_3'=\beta_2^2 \qquad \sigma_6'=2(\beta_2\beta_5+\beta_3\beta_4)$$
$$\sigma_4'=2\beta_2\beta_3 \qquad \sigma_7'=2(\beta_2\beta_6+\beta_3\beta_5)+\beta_4^2$$
$$\cdots\cdots\cdots\cdots$$

Here we have

$$\sigma_n = \sigma_n' + 2\beta_1\beta_n \qquad n > 1.$$

As an example let it be required to find the square root of π. We put down the outline of our work as follows:

	β_1	β_2	β_3	β_4	β_5
$\sqrt{\pi} =$					
$\pi = 031415926$	535897932	384626433	832795028	841971694	
γ_1	γ_2	γ_3	γ_4	γ	

In order to get β_1 we must first find the square root of γ_1 or

$$031415926 \quad 5.$$

By inspection we note that the square root lies between 17 and 18. Subtracting 17^2 from γ_1 and dividing the remainder by 17 we find the first digit in half the quotient to be 7. We therefore subtract 177^2 from γ_1 and by dividing the remainder by 177 and taking half the quotient we see that the next approximation to the root is 17724. One more application of this process gives $\beta_1 = 177245385$ and also $2\beta_1 = 345490770$, which last is written on the side for future use.

and also

We now set in the carriage the number γ_1, γ_2 from which we subtract β_1^2. This gives the remainder:

$$r_1 = 032099707 .$$

The number r_1, γ_3 or

$$032099707 \quad 384626433$$

is now divided by $2\beta_1$. This gives the following quotient and remainder:

$$q_2 = 090551602 \quad r_2 = 266912893 .$$

Squaring q_2 we obtain

$$q_2^2 = 008199592 \quad 624766404.$$

We observe that if q_2 were β_2 this last number would be σ_3'. Now it can be shown that σ_3' lies between r_2 and $r_2 - 2\beta_1$. It is seen that we have chosen r_2 correctly and that therefore $q_2 = \beta_2$. We may now put down β_1, β_2, σ_1, and σ_2 for $\sigma_1 = \beta_1^2$ and $\sigma_2 = 2\beta_1\beta_2$. The work then appears as follows:

	β_1	β_2	β_3	β_4	β_5
$\sqrt{\pi} = 177245385$	090551602				
031415926	503798225				
	032099707	117713540			
$\pi = 031415926$	535897932	384626433	832795028	841971694	
γ_1	γ_2	γ_3	γ_4	γ_5	

Adding the two periods common to σ_1 and σ_2 we get

$$535897932$$

which is γ_2 which serves as a check. We also double β_2 obtaining $2\beta_2 = 181103204$ which is set down for future use.

We wish now to determine β_3. The number r_2, γ_4 is set in the carriage and σ_2' subtracted from it. The remainder thus obtained is divided by $2\beta_1$ producing the following quotient and remainder:

$$q_3 = 729816748$$
$$r_3 = 250612664 .$$

We next calculate $2\beta_2 q_3$ taking care to set $2\beta_2$ in the key-board. This gives:

$$2\beta_2 q_3 = 132172151 \quad 395660592 .$$

The condition that this number which is an approximation at least to σ_4' shall lie between r_3 and $r_3-2\beta_1$ is satisfied by the r_3 chosen above. We therefore conclude that $q_3=\beta_3$ and that the above two-period number is σ_4'. We are now in a position to calculate σ_3 by adding $2\beta_1\beta_3$ to the value obtained for σ_3'. This being done, the work appears as follows:

β_1	β_2	β_3	β_4	β_5
$\sqrt{\pi}=177245385$	090551602	729816748		
031415926	503798225	266912893	582182364	
	032099707	117713540		
$\pi=031415926$	535897932	384626433	832795028	814971694
γ_1	γ_2	γ_3	γ_4	γ_5

Again adding the two periods directly above γ_3 we get

384626433

which is γ_3 itself.

If we wish only 45 significant figures in the square root of π the last two β's can be obtained in one step as follows.

As usual we set the number r_3, γ_3 in the carriage and subtract σ_4'. Dividing the remainder by $2\beta_1$ we try:

$$q_4=334114518 \quad r_4=692312242.$$

We next calculate $\beta_4+2\beta_3 q_4$ remembering to set $2\beta_3$ in the key-board. This gives:

593141695 374011176.

It is seen that we have made a fortunate choice of q_4 since this last number lies between r_4 and $r_4-2\beta_1$. We therefore have:

$$\beta_4=334114518$$
$$\sigma_5'=593141695 \quad 374011176.$$

We next find that $r_4-\sigma_5'$ when divided by $2\beta_2$ gives:

$$q_5=279754946 \quad r_5=407140404\ .$$

From this we assume that $q_5=\beta_5$ with doubt only as to the last figure. Knowing β_4 and β_5 we may find σ_4 by adding $2\beta_1\beta_4$ to σ_4' and find σ_5 by adding $2\beta_1\beta_5$ to σ_5'. Inserting these results in the work we finally have:

β_1	β_2	β_3	β_4	β_5
$\sqrt{\pi}=177245385$	090551602	729816748	334114518	279754946
031415926	503798225	266912893	582182364	692312242
	032099707	117713540	250612664	149659452
$\pi=031415926$	535897932	384626433	832795028	841971694
γ_1	γ_2	γ_3	γ_4	γ_5

Whence

$$\sqrt{\pi}=1.77245\ 38509\ 05516\ 02729\ 81674\ 83341\ 14518\ 27975\ 4946.$$

The above general process will serve to extract the square root of a number to any degree of accuracy. Simpler methods can be readily derived in which it is only necessary to write down the root, period after period. In fact when the number whose square root is to be extracted is an integer $<10^{10}$, sixty digits in the root can be found in less than five minutes time. These short methods are not satisfactory for extending indefinitely the square root of any number. Moreover we are obliged to dispense with the valuable checks to be had in the more general method. Similar statements may be made with regard to the

division process. We therefore have not given a detailed account of these short methods trusting that the reader can work them out for himself.

COMMUTATIVE ALGEBRAIC INVERSIONS

By E. T. BELL, University of Washington

1. Introduction. In arithmetic numerous interesting theorems concerning functions of divisors depend ultimately upon the inversion of Dedekind (simultaneously stated by Liouville), which is as follows. Let $p(x)$, $q(x)$ be single valued for integer values of x, and write $\mu(n)=0, 1, -1$ according as the integer $n>0$ is divisible by an integer square >1, or the number of prime factors in n is even or odd respectively; $\mu(1)=1$. Then

$$p(n)=\sum q(d) \text{ implies } q(n)=\sum \mu(d)p(n/d) , \tag{1}$$

the summation extending to all divisors $d>0$ of n. Simple proofs of (1) are given in most books on arithmetic; a summary of allied results will be found in Dickson's *History of the Theory of Numbers*, vol. 1, chap. 19.

It was observed by H. F. Baker[1] that (1) is included in a much wider theorem concerning symmetric functions.

Baker's inversion was in turn generalized by L. Gegenbauer.[2] These extensions rapidly become extremely complicated, at least in algebraic expression, and further generalization by ordinary algebra is typographically impracticable. All, however, not only become intelligible at a glance when the appropriate algorithm is introduced, but also admit of indefinite extension in similar or in essentially new directions with equal simplicity. By means of the elementary processes of this algorithm the theorems become self-evident.

2. Notation. Each of $f_i, g_i, \cdots, h_i$ $(i=1, 2, \cdots)$ denotes a symmetric function of precisely i variables chosen from the sets

$$a_1, a_2 , \cdots ; \quad b_1, b_2 , \cdots ; \cdots ; c_1, c_2 , \cdots .$$

It is essential to remember that by this notation the number of variables in any $f_i, g_i, \cdots, h_i$ is equal to the suffix i. Thus we may have $f_1(a_1), f_1(a_2), \cdots, f_2(a_1, a_2), \cdots$ but not $f_3(a_1, a_2)$. Further, the symmetric functions in a given context may be any whatever. For example, our theorems hold for $f_2(a_1, a_2) = a_1^2+a_2^2$, $f_3(a_1, a_2, a_3)=a_1^2+a_2^2+a_3^2$, or for any other symmetric functions of a_1, a_2 and a_1, a_2, a_3 respectively, the point being that different suffixes do not neces-

[1] *Proceedings of the London Mathematical Society*, vol. 21 (1889) pp. 30–32. *Cf.* Dickson, *loc. cit.* pp. 443–4, where the misprints of the original article are rectified. Baker's Theorem is not essentially different from our first example in §5.

[2] *Sitzungsberichte der kaiserlichen Akademie der Wissenschaften, Wien*, vol. 102 (1893) part II, pp. 951–978. Gegenbauer's most general result is included in our second example of §5.

sarily imply that the corresponding symmetric functions are of different *kinds* (in the example both *may* be sums of squares). When convenient the variables will be omitted, thus $f_3(a_1, a_2, a_3) = f_3$.

3. Composition. We now define a process which has the abstract properties of algebraic multiplication. Raise the suffix of each function considered, thus $f_i \equiv f^i$, $g_i \equiv g^i$, etc., so that, for example, $f_2(a_1, a_2) \equiv f^2(a_1, a_2)$ symbolically. Composition is defined by the equations, which obviously are consistent,

$$f^r(a_1, \cdots a_r) f^1(a_{r+1}) = f^{r+1}(a_1, \cdots, a_r, a_{r+1}) , \tag{2}$$

$$f^r(a_1, \cdots, a_r) g^1(a_{r+1}) = f_r(a_1, \cdots, a_r) g_1(a_{r+1}) , \tag{3}$$

$$\begin{aligned} &\{F^r(a_1, \cdots, a_r) + G^s(b_1, \cdots, b_s)\} H^t(c_1, \cdots, c_t) \\ &= F^r(a_1, \cdots, a_r) H^t(c_1, \cdots, c_t) + G^s(b_1, \cdots, b_s) \\ &\qquad \cdot H^t(c_1, \cdots, c_t) , \end{aligned} \tag{4}$$

in which F, G, H may be any of $f, g, \cdots, h$. If $f, g, \cdots$ occur with coefficients, as αf^r, βf^s, the $\alpha, \beta, \cdots$ being numbers of any field, the laws of composition are the same as in ordinary algebra, thus

$$\alpha f^1(a_1) \beta f^2(a_2, a_3) = \alpha\beta f_3(a_1, a_2, a_3) .$$

Since $f_r, g_r, \cdots, h_r$ are symmetric functions, it follows that composition is commutative and associative. By (4) it is also distributive. Hence composition and algebraic multiplication are abstractly identical.

It follows immediately from the definitions that each of

$$F^r + G^r = H^r , \qquad G^r = H^r - F^r , \qquad F^r = H^r - G^r$$

implies both of the others. This simple remark, as we shall see, is at the root of the inversions described in §1.

To illustrate composition we have

$$\begin{aligned} &\{f^1(a_1) + g^1(a_1)\} \{f^1(a_2) + g^1(a_2)\} \\ &\qquad = f_2(a_1, a_2) + f_1(a_1) g_1(a_2) + f_1(a_2) g_1(a_1) + g_2(a_1, a_2) . \end{aligned}$$

Clearly not every sum of symmetric functions can be decomposed in this manner into symbolic factors; it is only those that can that yield inversion theorems of the kinds mentioned.

4. Decomposition. This process can be considered as an inverse of composition. Let

$$p(x, y, \cdots, z) \equiv \sum_{a, b, \cdots, c} a_{a, b, \cdots, c} x^a y^b \cdots z^c$$

be a polynomial (not necessarily homogeneous) in $x, y, \cdots, z$. Replace $x, y, \cdots, z$ respectively by the $F, G, \cdots, H$ of §3 and denote the result by S_i,

where j is the total number of variables occurring as arguments in $F, G, \cdots, H$. We have then

$$S_j = \sum_{a,b,\cdots c,} a_{a,b\cdots,c} F_a G_b \cdots H_c \qquad (5)$$

as the *definition* of S_j, or, symbolically,

$$S_j = p(F, G, \cdots, H) . \qquad (6)$$

In the right of (5) a particular symbol of symmetric functions, say F, appears with various suffixes, say $a', a'', \cdots$, so that F accounts for symmetric functions having (*cf.* §2) precisely a', or a'', or, $\cdots$ variables. Let n denote an arbitrary constant integer. Then *any* determination of F_n, having n variables, from (6) in terms of functions $S, G, \cdots, H$ (all of those in (6) except F), is called *a decomposition* of (6) or of $p(F, G, \cdots, H)$ with respect to F. Similarly with respect to any one of $G, \cdots, H$.

Note that S_j as defined by (5) or (6) is not necessarily a symmetric function in *all* of its j variables; it is symmetric *separately* in the members of each set of variables occurring as arguments of functions F, or functions $G, \cdots$ respectively. In the illustration of §5 we discuss examples of both cases, S symmetric, S not symmetric, in all of its variables.

In what precedes we have restricted p to be a polynomial. This restriction is inessential; removing it, and making the few obvious changes requisite, we can extend the definition of decomposition to any function p which has a multiple power series expansion. For brevity it may here be assumed that the original series for p and those for each of its decompositions are absolutely convergent in the same region. It is both feasible and profitable, however, to discuss the decomposition of infinite series independently of their convergence, and to devise an interpretation of the results which shall be self consistent. We merely indicate this last possibility; the means for carrying it out are fully discussed in a former paper.[1]

Any set of decompositions of $p(F, G, \cdots, H)$ with respect to $F, G, \cdots, H$ respectively, will be called an *inversion* of $p(F, G, \cdots, H)$ or of (6).

It is not assumed for p general either that decompositions exist or that, if existing, they are unique. If $p(x, y, \cdots, z)$ can be resolved into a product of linear factors with coefficients in any domain of rationality, inversion is always possible and unique in the same domain, as appears from the definitions and the examples next given.

5. Inversions. (I) It will suffice to discuss three of the simplest. Let the variables $a, b, \cdots, c$ be s in number, and write

$$1 + f^1(x) = g^1(x) \qquad (x = a, b, \cdots, c) . \qquad (7)$$

[1] *Transactions of the American Mathematical Society*, vol. 25 (1923) pp. 135–154.

Then, by composition, we obtain

$$\{1+f^1(a)\}\{1+f^1(b)\}\cdots\{1+f^1(c)\}=g_s(a,b,\cdots,c) \qquad (s=1,2,\cdots). \tag{8}$$

The structure of the symmetric functions on the right is obvious from the left, if we imagine the symbolic product first distributed before all exponents are degraded to suffixes. To decompose (8) with respect to f we have from (7),

$$f^1(x)=g^1(x)-1 \qquad (x=a,b,\cdots,c), \tag{9}$$

and hence by composition,

$$f_s(a,b,\cdots,c)=(-1)^s\{1-g^1(a)\}\{1-g^1(b)\}\cdots\{1-g^1(c)\} \qquad (s=1,2,\cdots) \tag{10}$$

as an inversion of (8).

To see the meaning of the theorem expressed by (8), (10), take the case of 2 variables. Then, by (8),

$$g_2(a,b)=1+f_1(a)+f_1(b)+f_2(a,b),$$
$$g_1(a)=1+f_1(a), \qquad g_1(b)=1+f_1(b),$$

and (10) asserts that for these g's,

$$f_2(a,b)=1-g_1(a)-g_1(b)+g_2(a,b),$$

which is correct. For 3 variables the g's given by (8) are

$$g_3(a,b,c)=1+f_1(a)+f_1(b)+f_1(c)+f_2(a,b)+f_2(a,c)+f_2(b,c)+f_3(a,b,c),$$
$$g_2(a,b)=1+f_1(a)+f_1(b)+f_2(a,b),$$
$$g_2(a,c)=1+f_1(a)+f_1(c)+f_2(a,c),$$
$$g_2(b,c)=1+f_1(b)+f_1(c)+f_2(b,c),$$
$$g_1(a)=1+f_1(a), \qquad g_1(b)=1+f_1(b), \qquad g_1(c)=1+f_1(c),$$

and (10) is

$$f_3(a,b,c)=-1+g_1(a)+g_1(b)+g_1(c)-g_2(a,b)-g_2(a,c)-g_2(b,c)+g_3(a,b,c),$$

which is verified by inspection for the foregoing g's.

(II) As a second example we shall take precisely r classes of symmetric functions $f_s, g_s, \cdots, h_s$ $(s=1, 2, \cdots)$, the symbols $f, g, \cdots, h$ being r in number, and decompose

$$\{f^1(a)+g^1(a)+\cdots+h^1(a)\}\{f^1(b)+g^1(b)+\cdots+h^1(b)\}\cdots\{f^1(c)+g^1(c)+\cdots+h^1(c)\} \tag{11}$$
$$=k_s(a,b,\cdots,c) \qquad (s=1,2,\cdots)$$

with respect to f. As before, the structure of the functions k_s, which evidently by the left of (11) are symmetric in the s variables $a, b, \cdots, c$, is evident for any value of s if we think of the distributed form of the left. In particular, when $s=1$, we have $f^1(x)+g^1(x)+\cdots+h^1(x)=k^1(x)$ $(x=a, b, \cdots, c)$, and therefore as before, for $s=1, 2, \cdots$, we obtain

$$f_s(a,b,\cdots c)=(-1)^s\{g^1(a)+\cdots+h^1(a)-k^1(a)\}\{g^1(b)+\cdots+h^1(b)-k^1(b)\}\cdots\{g^1(c)+\cdots+h^1(c)-k^1(c)\} \tag{12}$$

as a decomposition of (11) with respect to f. Similarly for $g, \cdots, h$. Taken together these r decompositions furnish an inversion of (11).

To verify a formula such as (12) after reduction to non-symbolic form, we eliminate from the decomposition, here (12), that class of functions, here the k_s $(s=1, 2, \cdots)$, which was defined by the original composition, here (11).

(III) Consider r symbols $f, g, \cdots, h$ of symmetric functions, and s symbols $a, b, \cdots, c$ of variables, so that each of the sets

$$\sigma_i\equiv a_i, b_i, \cdots, c_i \qquad (i=1, \cdots, r)$$

contains precisely s variables. Clearly, if the s functions

$$f^1(x_1)+g^1(x_2)+\cdots+h^1(x_r) \qquad (x=a, b, \cdots, c)$$

be compounded, the result, F_{rs}, is not symmetric in all the variables, but is symmetric separately in the variables of each of the sets σ_i $(i=1, \cdots, r)$. Write

$$F_{rs}(a_1,b_1, \cdots ,c_1 ; a_2,b_2, \cdots ,c_2 ; \cdots ; a_r,b_r, \cdots ,c_r) = \prod_{x=a,\, b, \cdots,\, c} \{ f^1(x_1)+g^1(x_2)+ \cdots +h^1(x_r) \} , \tag{13}$$

so that

$$F^r(x_1 ; x_2 ; \cdots ; x_r)=f^1(x_1)+g^1(x_2)+ \cdots +h^1(x_r) \qquad (x=a,b, \cdots ,c)$$

and hence

$$f_1(x_1)=F^r(x_1 ; x_2 ; \cdots ; x_r)-g^1(x_2)- \cdots -h^1(x_r) .$$

The decomposition of (13) with respect to f is therefore

$$f_s(a_1,b_1, \cdots ,c_1)= \prod_{x=a,\, b, \cdots,\, c} \{F^r(x_1 ; x_2 ; \cdots ; x_r)-g^1(x_2)- \cdots -h^1(x_r)\} , \tag{14}$$

the right of which obviously is symmetric in $a_1, b_1, \cdots, c_1$, as it should be by inspection of the left. Similarly for the decompositions of (13) with respect to $g, \cdots, h$, which can be written down from (14) by suitable changes of suffixes, etc. Thus for g the left of (14) becomes $g_s(a_2, b_2, \cdots, c_2)$.

6. Glancing back we see that the imposition of symmetry upon the functions inverted is necessary in order to preserve commutativity in composition. It is clear that this analogue of multiplication when applied to non-symmetric algebraic functions no longer is commutative. The composition of a symmetric function is isomorphic in its abstract properties with the resolution of an integer into its prime factors. All of the processes are reversible.

QUESTIONS AND DISCUSSIONS

EDITED BY TOMLINSON FORT, Hunter College, Park Ave. and 68th St., New York, N. Y., and H. E. BUCHANAN, Tulane University, New Orleans, La.

The Department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

REPLIES TO QUESTIONS

55 [*1925*, 506, 510]. Is it possible by ruler and compasses to construct an angle equal to one radian?

REPLY BY A. J. KEMPNER, University of Colorado.

Of course not. If an angle of one radian could be constructed by ruler and compasses, sin 1, cos 1, $\cos 1+i \sin 1=e^i$ could each be constructed, and e^i would be an algebraic number. But e^x is transcendental for all algebraic exponents $x \neq 0$, by a classical theorem.

NOTE: The proposer of the question informs us that he received replies from "the ends of the earth." EDITOR.

DISCUSSIONS

F. van Schooten

I. A Cubic Equation of Newton's

By Norman Anning, University of Michigan

1. History. In chapter 13 of his *Arithmetica Universalis* Newton showed how algebra could be applied to the solution of problems in geometry. He used for illustration the problem[1] of finding the diameter $AD(=x)$ of a semicircle when three chords AB $(=a)$, BC $(=b)$, CD $(=c)$ are given. In half a dozen interesting ways he arrived at the equation:

$$f(x)\equiv x^3-x(a^2+b^2+c^2)-2abc=0\ . \tag{1}$$

Having derived the equation he did not concern himself with its solution or discussion. He said[2] in conclusion

"Wherefore if the Method you take from your first Thoughts, for solving a Problem, be but ill accommodated to computation, you must again consider the Relations of the Lines, until you shall have hit on a Way as fit and elegant as possible. For those Ways that offer themselves at first Sight, may often create sufficient Trouble if they are made use of."

2. Nature of the roots.[3] If we neglect a positive numerical factor, the discriminant of (1) is $$(a^2+b^2+c^2)^3-27a^2b^2c^2\ . \tag{2}$$

When a, b, c, are real and not all equal, the three numbers a^2, b^2, c^2, are positive and not all equal. For such numbers

$$A.M.>G.M.,\qquad (a^2+b^2+c^2)/3>(a^2b^2c^2)^{1/3},\qquad (a^2+b^2+c^2)^3>27a^2b^2c^2\ .$$

In this case the discriminant (2) is positive and the cubic has three real and distinct roots.

Of the three roots one will be positive when a, b, c, are all positive and two will be positive when any one of a, b, c, is negative. For, in the former case, $f(\infty)$ is positive while $f(0)$ is negative and therefore $f(x)=0$ has an odd number of positive roots. But, by Descartes' theorem, it cannot have more than one. Consequently there is just one. The second case may be proved similarly or by observing that substituting $(-c)$ for c in (1) transforms $f(x)$ into $f(-x)$.

When $a=b$, the roots of (1) are constructible because then

$$x^3-(2a^2+c^2)x-2a^2c=(x+c)(x^2-cx-2a^2)=0\ .$$

When $a=b=c$, the discriminant is zero, and x has the values: $-a, -a, 2a$.

---

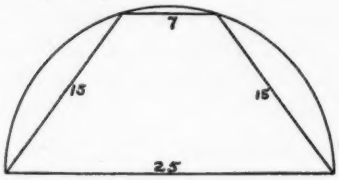
[1] See S. Horsley, *Isaaci Newtoni Opera quae exstant omnia*, London, 1779, vol. I, pp. 83–92. For a translation into English see Ralphson and Cunn, *Universal Arithmetick*, London, 1769, pp. 202–216.

[2] Quoted from Ralphson and Cunn, p. 216.

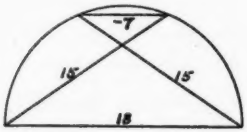
[3] Some of the results in this section were proved in *School Science and Mathematics*, 1911, problems 250, 254.

3. Numerical example.

When $a=b=15$, $c=7$,
$x^3-499x-3150=0$,
$x=25,-18,-7$.



When $a=b=15$, $c=-7$,
$x^3-499x+3150=0$,
$x=-25,18,7$.



This example suggests that two of the roots of (1) are capable of geometric interpretation. This will always be true provided (1) has two roots which are, in absolute value, not less than the greatest of the absolute values of a, b, c.

4. Solution in integers. If we write (1) in the form

$$\left(\frac{a}{x}\right)^2+\left(\frac{b}{x}\right)^2+\left(\frac{c}{x}\right)^2+2\frac{abc}{xxx}=1\ ,$$

and compare it with the identity $\cos^2A+\cos^2B+\cos^2C+2\cos A\cos B\cos C=1$, which is true whenever $A+B+C=180°$, we see that the problem[1] of finding integers a, b, c, x, to satisfy (1) reduces to that of finding a triangle whose angles have rational cosines. The sides of such a triangle must all be rational multiples of the same quantity and by suitably changing the scale we can, without altering the angles, make the measures of the sides all integers. So if we choose any three positive integers l, m, n, which are the sides of a possible triangle, we may write

$$a=l(m^2+n^2-l^2)\ ,\quad b=m(n^2+l^2-m^2)\ ,\quad c=n(l^2+m^2-n^2)\ ,\quad x=2lmn\ .$$

For instance, putting $l=4$, $m=5$, $n=6$, yields $a=12$, $b=9$, $c=2$, $x=16$.

II. Another Poristic System of Triangles[2]

By Rufus Crane, Ohio Wesleyan University

A triangle is, in general, determined by any two of its four in- and escribed circles. Again, it is determined by the circumcircle, the nine-point circle, and any one of the in- or escribed circles. In this latter case, if only two of the circles be fixed, the triangle has one degree of freedom and may assume a series of positions. These constitute a poristic system. Gallatly, in his *Modern Geometry of the Triangle*, 1913, discussed the poristic system with the incircle and the

[1] A solution of this problem by P. Bachmann was published in *Archiv der Mathematik und Physik*, series III, vol. 24 (1915), pp. 89–90.

[2] Read before the American Mathematical Society, Chicago meeting, April 10, 1925.

circumcircle fixed. Professor J. H. Weaver, in the MONTHLY, September, 1924, discussed the case where the circumcircle and the nine-point circle were fixed. The present note discusses the case of the nine-point circle and the incircle.

The radii of these two circles may have any values whatever, provided only that the incircle is not greater than the nine-point circle. If the incircle coincides with the nine-point circle, the triangle becomes equilateral and the system degenerates.

In the notation here used the circumcircle has center O, radius R; the nine-point circle has center O', radius $\frac{1}{2}R$; the incircle has center I, radius r, and touches BC at X, CA at Y, AB at Z; the excircle opposite A has center I_1, and touches BC at X_1, CA at Y_1, AB at Z_1; the excircle opposite B has center I_2, etc. The point of tangency of the incircle and nine-point circle is F, the Feuerbach point.

Since the distance OI is constant, in terms of the invariants R and r, the locus of O is a circle with center at I and this distance as radius. This circle will lie within, coincide with, or lie without the incircle according as

$$R \lesseqgtr r(1+\sqrt{2}) .$$

In the first case the triangle is acute angled in all of its positions; in the second case it becomes right angled when O falls at F; in the third case it is acute angled as long as O lies inside the nine-point circle, but is obtuse angled when O falls outside of, and right angled when it falls on that circle.

Obviously, the center of gravity G, the orthocenter H, and the point J (the circumcenter of $I_1I_2I_3$), traverse circles homothetic to that of O. Obviously, also, the Nagel point N traverses the same circle as H.

Aside from these, it may readily be shown that the Gergonne point M traverses a circle. Gallatly has shown that the point σ (the center of similitude of the triangles XYZ and $I_1I_2I_3$) and also the point H_i (the orthocenter of XYZ) are points lying on the line OI at distances from I which are expressible in terms of R and r. He also shows that σ, M, G, are collinear, as likewise H_i, M, N. Also, I, G, N, are collinear and IG is one third of IN. Hence, the lines NH_i and $G\sigma$ intersect the lines IO' and IO at distances which are invariant: hence, their point of intersection M has invariant coordinates referred to these two lines as oblique axes.

Some of the other points of the triangle give us loci of higher degree. The equations of these may be written out, with greater or less ease, by determining the distances of the point in question from some of the points O', I, F, or some other critical point, in terms of R, r, and the angles A, B, C, then taking one of these points as a pole and eliminating A, B, and C. The details of these derivations are routine, but the following results may be stated.

The locus of the vertices A, B, C, is a bicircular quartic, very similar to the nodal form of the circular limaçon. Taking the pole at the node, which lies on $O'I$ at a distance r from F, with $O'I$ as axis, its equation becomes

$$\rho=(R+2r)\cos\theta\pm\sqrt{(R-2r)^2\cos^2\theta+2r(R-2r)}\ .$$

This curve has one focus at I, and another external to the curve, at a distance from I equal to $8Rr/(R-2r)$. We may note that while the vertices are traversing this locus once, the points mentioned above traverse their loci three times.

The excenters I_1, I_2, I_3, also generate a bicircular quartic, in fact, a limaçon, with a focus at O', with a node on IO' distant $2R$ from I. Taking this node as origin, with IO' as axis, its equation is

$$\rho=4R\cos\theta\pm2\sqrt{R^2-2Rr}\ .$$

The nine points of tangency of the excircles fall in two groups; the three that lie on the sides AB, BC, CA, and which may be called the internal points of tangency, form one group, while the six that lie externally (on the sides extended) form another group. The first group, X_1, Y_2, Z_3, generate a curve of sixth degree passing twice through the Feuerbach point and having a tacnode at that point. The curve has also a crunode lying on the line FIO'. Depending on the relation between R and r mentioned above, the shape of this curve varies. If R is less than $r(1+\sqrt{2})$, the curve is convex throughout: if greater, the outer loop has a sinus at the tacnode. The equation of this curve (and likewise of the following) is too cumbersome to be of value except as showing the degree of the curve.

The six external points of tangency generate a locus consisting of two bicircular curves which are located obliquely to the line FIO', but are symmetrical to each other with respect to this line. This locus proves to be of degree sixteen.

III. A New Method of Determining Bernoulli's Numbers[1]

By C. A. Messick, University of Iowa

1. The methods of determining Bernoulli's numbers seem to group themselves into two general classes, one depending on differentiation,[2] and the other on multiplication of infinite series.[3]

---

[1] Presented to the American Mathematical Society at the meeting in Kansas City, December 29, 1925.

[2] Price, *Treatise on Infinitesimal Calculus*, p. 145.
Godefroy, *Théorie Elémentaire des Series*, p. 116.

[3] Bromwich, *Introduction to the Theory of Infinite Series*, p. 233.
Smail, *Elements of the Theory of Infinite Processes*, p. 238.

In the first method, successive differentiation of the function $F(x)=\frac{x}{e^x-1}$ produces the equations:

$$e^x[F'(x)+F(x)]=1+F'(x)\ ,$$

$$e^x\left[F^{(n)}(x)+nF^{(n-1)}(x)+\frac{n(n-1)}{2!}F^{(n-2)}(x)+\cdots+nF'(x)+F(x)\right]=F^{(n)}(x)\ ,$$

where $n=2,\ 3,\ 4,\ \cdots\cdots$. The nth Bernoulli number is defined by $B_n=(-1)^{n-1}[F^{(2n)}(0)]$, all the derivatives of odd order after the first being zero when $x=0$. The Maclaurin series is

$$F(x)=\frac{x}{e^x-1}=1-\frac{x}{2}+B_1\frac{x^2}{2!}-B_2\frac{x^4}{4!}+B_3\frac{x^6}{6!}-\cdots\ .$$

In the second method, it is assumed[1] that the function has the expansion $x/(e^x-1)=1+A_1x+A_2x^2+A_3x^3+\cdots$. It is found that all the odd numbered A's after the first are zeros, and the even numbered A's are determined by the equations

$$\frac{A_{2n}}{1}+\frac{A_{2n-2}}{3!}+\cdots+\frac{A_2}{(2n-1)!}-\frac{2n-1}{2(2n+2)!}=0\ ,$$

$$\frac{A_{2n}}{2!}+\frac{A_{2n-2}}{4!}+\cdots+\frac{A_2}{(2n)!}-\frac{2n}{2(2n+2)!}=0\ ,$$

where $n=1,\ 2,\ 3,\ \cdots$. By comparing with the Maclaurin series, it is seen that we can define the nth Bernoulli number as $B_n=(-1)^{n-1}(2n)!A_{2n}$.

By the methods of the first class a single equation is derived, and by the methods of the second class usually two equations are derived, each containing an arbitrary positive integer n. To find the rth Bernoulli number, set $n=r$ in the equation, substitute the numerical values of the first $r-1$ numbers and perform the indicated operations.

By the new method (which is of the second kind) given below, four equations are obtained which seem different from any given before. From one equation the Bernoulli numbers of odd index can be computed, and from another, the numbers of even index. The advantage in using these equations is that there are only half as many terms as in the equations now in use, so that the arithmetic is much more simple. The other two equations are sums of multiples of an odd or even number, respectively, of the Bernoulli numbers, and can be used to check the numerical accuracy of the results obtained from the first pair.

[1] This is justified by a theorem in Godefroy, p. 96, "If a function, here (e^x-1/x), can be developed into a power series in which the first term is not zero, its reciprocal can be developed into a power series in which the first term is not zero."

An expansion for $\sin x$ is also obtained in terms of Bernoulli's numbers, which is believed to be new. This is $F(x) \cdot \varphi(x)$, given below.

2. It is known that if $|x| < 1.2$,

$$F(x) = \frac{x}{e^x - 1} = 1 - \frac{x}{2} + B_1 \frac{x^2}{2!} - B_2 \frac{x^4}{4!} + B_3 \frac{x^6}{6!} - \cdots + (-1)^{n-1} B_n \frac{x^{2n}}{(2n)!} \cdots \tag{1}$$

where B_n represents the nth Bernoulli number.

Now let $\varphi(x)$ be $\dfrac{e^{(1+i)x} - e^{(1-i)x} - e^{ix} + e^{-ix}}{2ix}$.

Expanding this function into a power series in x, we have

$$\varphi(x) = x + \frac{3x^2}{3!} - \frac{5x^4}{5!} - \frac{8x^5}{6!} - \frac{7x^6}{7!} + \frac{15x^8}{9!} + \frac{32x^9}{10!} + \frac{33x^{10}}{11!} - \frac{65x^{12}}{13!} - \frac{128x^{13}}{14!} \cdots + (-1)^k \left[\frac{[2^{2k} - (-1)^k] x^{4k}}{(4k+1)!} + \frac{[2^{2k+1}] x^{4k+1}}{(4k+2)!} + \frac{[2^{2k+1} - (-1)^{k+1}] x^{4k+2}}{(4k+3)!} \right] \cdots$$

where $k = 0, 1, 2, 3, \cdots$. Since this power series is convergent in the interval of convergence of (1), we may multiply it by $F(x)$, obtaining a power series whose interval of convergence is at least as great as that of (1), that is, $|x| < 1.2$.

$$F(x) \cdot \varphi(x) = x - x^3 \left(\frac{1}{4} - \frac{B_1}{2} \right) + x^4 \left(\frac{B_1}{4} - \frac{1}{4!} \right) + x^5 \left(\frac{1}{2 \cdot 4!} - \frac{8}{6!} - \frac{B_2}{4!} \right) - x^6 \left(\frac{7}{7!} - \frac{4}{6!} + \frac{B_1}{2 \cdot 4!} + \frac{B_2}{2 \cdot 4!} \right) - x^7 \left(\frac{8B_1}{2 \cdot 6!} - \frac{7}{2 \cdot 7!} - \frac{B_3}{6!} \right) + x^8 \left(\frac{15}{9!} - \frac{7B_1}{2 \cdot 7!} + \frac{B_2}{4!^2} + \frac{B_3}{2 \cdot 6!} \right) + x^9 \left(\frac{32}{10!} - \frac{15}{2 \cdot 9!} + \frac{8B_2}{4! \cdot 6!} - \frac{B_4}{8!} \right) - \cdots \pm C_n x^n \cdots$$

where C_n is as given below.

But $F(x) \cdot \varphi(x) = \dfrac{e^{ix} - e^{-ix}}{2i} = \sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

which is convergent for all values of x. We may now equate the coefficients of like powers of x in the two series and compute the numerical values of B_1, B_2, $\cdots$ as far as we wish. This gives

$$B_1 = \frac{1}{6}, \qquad B_2 = \frac{1}{30}, \qquad B_3 = \frac{1}{42}, \qquad B_4 = \frac{1}{30}, \cdots.$$

The coefficient C_n of the nth power of x in the product $F(x) \cdot \varphi(x)$ takes four forms according to whether n is of the form $4k$, $4k+1$, $4k+2$, or $4k+3$, ($k = 1, 2, 3, \cdots$). The equation when n is odd is $C_n = 1/n!$, and when n is even, $C_n = 0$. When n is odd

$$C_n = -\frac{B_{\frac{n-1}{2}}}{(n-1)!} + \frac{B_{\frac{n-5}{2}}2^3}{(n-5)!6!} - \frac{B_{\frac{n-9}{2}}2^5}{(n-9)!10!} + \cdots$$

ending, when $n = 4k+1$ and B has only even subscripts, as follows:

$$\cdots + (-1)^k\left[\frac{B_{2}2^{2k-1}}{4!(4k-2)!} - \frac{1}{2}\,\frac{2^{2k}-(-1)^k}{(4k+1)} + \frac{2^{2k+1}}{(4k+2)!}\right],$$

ending, when $n = 4k+3$ and B has only odd subscripts, as follows:

$$\cdots + (-1)^{k+1}\left[\frac{B_{1}2^{2k+1}}{2!(4k+2)!} - \frac{1}{2}\,\frac{2^{2k+1}-(-1)^{k+1}}{(4k+3)!}\right].$$

When n is even

$$C_n = \frac{B_{\frac{n-2}{2}}(2^1+1)}{(n-2)!3!} + \frac{B_{\frac{n-4}{2}}(2^2+1)}{(n-4)!5!} - \frac{B_{\frac{n-6}{2}}(2^3-1)}{(n-6)!7!} - \frac{B_{\frac{n-8}{2}}(2^4-1)}{(n-8)!9!} + \cdots$$

ending when $n = 4k$ as follows:

$$\cdots + (-1)^k\left[\frac{B_{2}[2^{2k-2}-(-1)^{k-1}]}{4!(4k-3)!} - \frac{B_{1}[2^{2k-1}-(-1)^k]}{2!(4k-1)!} + \frac{2^{2k}-(-1)^k}{(4k+1)!}\right]$$

ending when $n = 4k+2$ as follows:

$$\cdots + (-1)^{k+1}\left[\frac{B_{2}[2^{2k-1}-(-1)^k]}{4!(4k-1)!} + \frac{B_{1}[2^{2k}-(-1)^k]}{2!(4k+1)!} - \frac{2^{2k}}{(4k+2)!} + \frac{2^{2k+1}-(-1)^{k+1}}{(4k+2)!}\right].$$

A Crystallographic Illustration of Quadratic Involution

By A. C. Lunn, University of Chicago

Let $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ be a triple of vectors restricted to be of equal magnitude and inclined at equal angles, so that in the Gibbs notation $\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = q$, $\mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b} = p$, where q, p are arbitrary except that for real vectors q is positive and p numerically less than q. Then it is apparent from the geometry, and readily provable from the definitions $v\mathbf{a}' = \mathbf{b} \times \mathbf{c}$, $v\mathbf{b}' = \mathbf{c} \times \mathbf{a}$, $v\mathbf{c}' = \mathbf{a} \times \mathbf{b}$, where $v = [\mathbf{abc}]$, that the "reciprocal triple" satisfies similar conditions, only

with a new pair of constants q', p'. One way to show this is to notice that the general resolution theorem, $\mathbf{r}=\mathbf{r}\cdot\mathbf{aa}'+\mathbf{r}\cdot\mathbf{bb}'+\mathbf{r}\cdot\mathbf{cc}'$, gives here

$$\mathbf{a}=q\mathbf{a}'+p(\mathbf{b}'+\mathbf{c}'),\quad \mathbf{b}=q\mathbf{b}'+p(\mathbf{c}'+\mathbf{a}'),\quad \mathbf{c}=q\mathbf{c}'+p(\mathbf{a}'+\mathbf{b}'),$$

whose solution proves to be of similar form, with primed and unprimed interchanged, if q', p' be defined by

$$p'=-p/r\ ,\qquad q'=(q+p)/r\qquad \text{with } r=(q-p)(q+2p)\ .$$

Since the relation between the two triples of vectors is mutual this relation must be an involution, and in fact the interchange of primed and unprimed does give the solution for p', q' in terms of p, q.

To put the result in a different form let x, x' be the cosines of the respective included angles, then $x=p/q$, $x'=p'/q'$, so that

$$qq'=\frac{1+x}{(1-x)(1+2x)}\ ,\qquad x'=-\frac{x}{1+x}\ .$$

Here the second equation must be an involution, and it is by change of sign of x seen to be one of the familiar cross-ratio transformations of period two; and the first expression, being symmetric in q, q', must be an invariant of that transformation.

Two cases occur in the theory of lattices belonging to the cubic system. One is the case of the common orthogonal triple $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ of the simple cube lattice, where the reciprocal triple is the same; here $p=p'=0$, $qq'=1$. The other case can be represented with its first triple $(\mathbf{j}+\mathbf{k}, \mathbf{k}+\mathbf{i}, \mathbf{i}+\mathbf{j})/2$, which generates a face-centered cube lattice, with reciprocal triple $(\mathbf{j}+\mathbf{k}-\mathbf{i}, \mathbf{k}+\mathbf{i}-\mathbf{j}, \mathbf{i}+\mathbf{j}-\mathbf{k})$, which gives the dual body-centered lattice; here $q=1/2$, $p=1/4$, $q'=3$, $p'=-1$; the cosines are $1/2$, $-1/3$, and the invariant $3/2$.

Various curious extensions of these relations are obtainable by considering n-tuples of vectors in higher spaces and symmetrizing their magnitudes and orientations in various ways.

On the Duals of Metric Theorems

By Louis Weisner, University of Rochester

In dualising theorems involving angular magnitudes we frequently invoke the

Theorem 1. *The angle between two lines not passing through the center of a circle is equal to the angle subtended at the center of the circle by the poles of the lines with respect to the circle.*

A few theorems[1] involving distances may be dualised with the aid of the

[1] See, for example, Winger, *Projective Geometry*, p. 148, ex. 21–23.

THEOREM 2. *The product of the distances of a point and its polar line from the center of a circle is constant, equal to the square of the radius of the circle.*

The limitations of this theorem being obvious, we proceed to prove a theorem which enables us to dualise such metric theorems as the Pythagorean theorem.

THEOREM 3. *If P_1, P_2 are the feet of the perpendiculars from the center O of a circle, upon the polar lines of two points A_1, A_2 (different from O) with respect to the circle, then $\overline{A_1A_2}=\overline{P_1P_2}a^2/\overline{OP_1}\cdot\overline{OP_2}$, where a is the radius of the circle.*

The theorem follows from the fact that P_1, P_2 are the inverses of A_1, A_2 with respect to the circle, and from the well-known formula for the distance between the inverses of two points.

The following illustrations exhibit the applications of Theorem 3. The usual machinery of reciprocation is freely used.[1]

1. The dual of the Pythagorean theorem is: If P_1, P_2, P_3 are the feet of the perpendiculars on the sides opposite the vertices A_1, A_2, A_3 of a triangle, from a point O on the circle whose diameter is A_2A_3, then

$$\left(\frac{\overline{P_2P_3}}{\overline{OP_2}\cdot\overline{OP_3}}\right)^2=\left(\frac{\overline{P_1P_2}}{\overline{OP_1}\cdot\overline{OP_2}}\right)^2+\left(\frac{\overline{P_1P_3}}{\overline{OP_1}\cdot\overline{OP_3}}\right)^2.$$

2. The reciprocal of a pair of triangles whose sides are respectively parallel is a pair of triangles perspective from the center of reciprocation. From the properties of similar triangles we may derive properties of perspective triangles. For example, since the homologous sides of similar triangles are proportional, we have the following theorem: Let the triangles $A_1A_2A_3$ and $A_1'A_2'A_3'$ be perspective from a point O, the points O, A_i, A_i' $(i=1, 2, 3)$ being collinear. If P_i and P_i' are the feet of the perpendiculars from O on the sides opposite A_i and A_i' respectively, then $(\overline{P_1P_2}/\overline{P_1P_3})\cdot(\overline{OP_3}/\overline{OP_2})=$ the same function of the accented letters, with $O'=O$. This function is therefore an invariant as regards all triangles perspective from O with triangle $A_1A_2A_3$.

3. From the theorem "the tangents to a circle from an external point are equal," we deduce that, if P_1, P_2, P_3 respectively are the feet of the perpendiculars from the focus O of a conic, on two arbitrary tangents and the chord joining the point of contact, then

$$\frac{\overline{P_1P_3}}{\overline{OP_1}}=\frac{\overline{P_2P_3}}{\overline{OP_2}}.$$

[1] An excellent exposition of the subject of reciprocation, a knowledge of which is essential for the comprehension of the illustrations given here, will be found in Winger, *Projective Geometry*, p. 138ff.

4. From Ptolemy's theorem on quadrilaterals, we deduce that if P_1, P_2, P_3, P_4, in cyclic order, are the feet of the perpendiculars from the focus O of a conic, on four arbitrary tangents, then

$$\frac{\overline{P_1P_3}}{\overline{OP_1}\cdot\overline{OP_3}}\cdot\frac{\overline{P_2P_4}}{\overline{OP_2}\cdot\overline{OP_4}}=\frac{\overline{P_2P_3}}{\overline{OP_2}\cdot\overline{OP_3}}\cdot\frac{\overline{P_1P_4}}{\overline{OP_1}\cdot\overline{OP_4}}+\frac{\overline{P_1P_2}}{\overline{OP_1}\cdot\overline{OP_2}}\cdot\frac{\overline{P_3P_4}}{\overline{OP_3}\cdot\overline{OP_4}}.$$

Clearing of fractions we have an equation which proves that P_1, P_2, P_3, P_4 are either collinear or concyclic. Hence, the pedal of a conic with respect to a focus is a line or a circle. (It is a line if the conic is a parabola.)

VI. A Note on the Hindu-Arabic Numerals

By Bibhutibhusan Datta, University College of Science and Technology, Calcutta

In Professor Smith's *History of Mathematics* (vol. II, p. 64) and in the work on the *Hindu-Arabic Numerals* written by him and Professor Karpinski, due credit is given to the evidence of the Hindu origin of these forms. In the latter work, however, it appears (p. 41) that the authors entertain the belief that, although the place value was invented in the sixth century A.D., "not until a considerably later period did it become well known."

It is certain, however, that the decimal system of numeration was familiar to the Vedic Hindus in almost the same form as at present. There are references in Indian literature which prove conclusively that the idea of place value was well known in India in the fifth century A.D., if not a century or two earlier. Fortunately, too, the references are not merely epigraphical, nor are they of doubtful value. One such reference is found in Śaṅkarāchārya's commentary on the Brahma-Sūtra, and also in Vyāsa's commentary on the Yoga-Sūtra of Patañjali. Commenting on Chapter II, Pāda 2, Sūtra 17, Śaṅkarāchārya writes:

"As though the line[1] is one and the same, but being placed in different positions, it indicates the ideas denoted by the different words 'one,' 'ten,' 'hundred,' 'thousand,' etc., so · · · ".

The translation is literal, the text is simple and explicit on the subject of place value, and it cannot, considering the context, be interpreted otherwise.

The passage in the commentary on the Yoga-Sūtra is almost identical and has been translated by Professor Wood as follows[2]:

---

[1] See footnote 3.

[2] *Vide* the *Yoga-System of Patañjali*, translated by Professor J. H. Wood (*Harvard Oriental Series*) p. 216. In the footnote Professor Wood has noted the significance of this passage as indicative of the knowledge of place value amongst the ancient Hindus. *Vide* also Sarkar, *Hindu Achievement in Exact Science* (London).

"Thus the same stroke[1] is termed one in the unit-place and ten in the ten's place and a hundred in the hundred's place."

This reference affords, therefore, the most conclusive evidence that both Vyāsa and Śaṅkarāchārya were acquainted with the significance of the place value.

Now, Śaṅkarāchārya was not a mathematician; he was a triumphant Hindu religious reformer. He travelled on foot throughout the greater part of India, giving a new interpretation to the Ancient Śāstras, especially the Brahma-Sūtra, challenging the savants to philosophical controversies, defeating them, and bringing them into his fold. He wrote his commentary on the Brahma-Sūtra to help his preaching, so it may be safely assumed without any fear of controversion that the idea of place value was generally known to the élite of his time. If this were not the case, and if the subject did not form a part of the elementary training of his boyhood, how could a philosopher like Śaṅkarāchārya, who took up the life of a secluded ascetic at the early age of eight, not becoming a religious reformer and preacher until a later period, know of one of the greatest of human inventions in a branch of knowledge so different from his own? It is enjoined in Kautilya's Arthaśāstra (book I, chapter V), a book written in the beginning of the fourth century B.C., that "having undergone the ceremony of tonsure" (that is, at about the age of five), "the student shall learn the alphabet (*lipi*) and arithmetic" (*sankhyanam*, literally the science of numerals). The usual rule in such cases is to assume that the idea in question must have originated about a century or two previous to the epoch of the document in which it appears.

The question then arises as to the time in which Śaṅkarāchārya lived, a matter upon which scholars differ widely. Orthodox Hindu writers, following the chronology preserved in the archives of the Muṭhas (monasteries) founded by Śaṅkarāchārya, believed that he flourished in the beginning of the first century B.C. Amongst other oriental scholars, however, there are some (like Telang) who are of opinion that he lived in the latter part of the sixth century A.D., while others (like Pathak, followed by Max Müller, and Wilson) put him in the eighth century and still others place him in the seventh century. It is also noteworthy that amidst the latter group Tilak and Ghose have come to nearly concurrent dates by proceeding on different lines. The date of Vyāsa's commentary on the Yoga-Sūtra of Patañjali cannot be later than the sixth century A.D. So in any case it seems safe to assume that the place value was well known in India in the fifth century A.D., and probably earlier.

[1] The original sanskrit text in either case is 'rekhā' which may mean 'line,' as also 'numeral'. I have translated it as 'line' on the idea that the first numeral was denoted by a simple line, a stroke, about the same age as is found in ancient inscriptions. Prof. Wood translates it by 'stroke.' It may be further noted that the word 'rekhā' comes from the same root as the word 'lekhā' which means 'writing.'

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

The Mechanical Investigations of Leonardo Da Vinci. BY I. B. HART. Chicago, The Open Court Publishing Company, 1925. 240+vi pages, 4 plates, 136 figures in text. Price $4.00.

Preliminary to a detailed study of Leonardo's aeronautical investigations, which is the chief object of this volume, the writer devotes a chapter each to a rather minute account of the state of mechanical science during the first half of the fifteenth century, the contemporary scientific influences which bore upon Leonardo during his life time, and the known sources of his amazing knowledge. These are followed by two chapters on Leonardo's mechanics, the first dealing with his contributions to dynamics, the second with his contributions to statics. The substance of the last of these chapters has previously appeared in the *Open Court.*

These five chapters, together with the introductory chapter on the characteristic of Leonardo's manuscripts, cover the first 143 of the 240 pages of the book. This part of the work is admirably done and while it contains little that may not be found in the earlier writers, such as Duhem and Seailles in France and Werner, Grothe, and Feldhaus in Germany, it serves well the purpose of a brief history of the state of progress of the mechanical sciences during the fifteenth century. Some knowledge of the limitations of the investigators of his time is quite indispensable to a fair appraisal of Leonardo's achievements in the field of aeronautics.

The seventh chapter which deals with Leonardo as a pioneer of aviation appeared as a separate paper in the *Journal of the Royal Aeronautical Society.* Here the author brings together and views the passages from Leonardo's various note books which bear on the subject of artificial flight. The treatment cannot be called exhaustive for we notice the omission and lack of reference, among others, to such important utterances relating to flight as the following:[1]

"Dissect the bat, study it carefully, and on this model construct the machine (Ms. F 41 v.).

The bird in its flight without the help of the wind drops half the wing downwards, and thrusts the other half toward the tip backwards; and the part which is moved down prevents the descent of the bird, and that which goes backwards drives the bird forwards.

When the bird raises its wings it brings their extremities near together; and while lowering them it spreads them further apart during the first half of the movement, but after this middle stage, as they continue to descend it brings them together again. (Ms. K 12 v.).

When the bird lowers one of its wings necessity constrains it instantly to extend it, for if it did not do so it would turn right over. (Ms. K 4 v.).

There are many birds which move their wings as swiftly when they raise them as when they let them fall: such as magpies and birds like these.

[1] The renderings into English are from McCurdy's *Leonardo da Vinci's Notebooks*, New York, 1923.

There are some birds which are in the habit of moving their wings more swiftly when they lower them than when they raise them, and this is seen to be the case with doves and such birds.

There are others which lower their wings more slowly than they raise them, and this is seen with rooks and other birds like these. (Ms. L 58 v., 59 v.)

The speed of birds is checked by the opening and spreading out of the tail. (Ms. L 59 v.)

The thrushes and other small birds are able to make headway against the course of the wind, because they fly in spurts; that is they take a long course below the wind, by dropping in a slanting direction towards the ground, with their wings half closed, and they then open the wings and catch the wind in them with their reverse movement, and so rise to a height; and then they drop again in the same way. (Ms. C.A. 313 r.b.)"

On the other hand there are some unnecessary repetitions as, for instance, the quotation consisting of 13 lines from Ms. K folio 10 v. which appears in full in two places, pp. 161–162 and pp. 178–179 and the shorter quotation from C.A. fol. 161 r.a. which is repeated on successive pages, p. 156, p. 157.

Notwithstanding these shortcomings this chapter offers the most complete treatment of its kind which has appeared up to the present time. Enough is said to show that Leonardo approached the problem of flight in a truly scientific spirit and that he anticipated, not always by very conclusive reasoning, to be sure, many of the accepted principles of aeronautical science. In this as in so many other fields, Leonardo penetrates below the surface of things and lays hold of essentials in a way which, as Hallam remarked long ago, strikes one with something like the "awe of preternatural knowledge."

The final chapter of the book consists of a translation of Leonardo's manuscript known as the *Codice sul Volo degli Uccelli e Varie Altre Materie* (Codex on the Flight of Birds and other Matters), together with a short history of the manuscript itself. This in the reviewer's opinion is perhaps the most valuable part of the book, since it is the first complete translation into English, though not the first publication, since like Chapter VII, this chapter has previously appeared as a separate paper in the *Journal of the Royal Aeronautical Society*. The term "complete" used in the preceding sentence needs, however, some modification. It is complete as to text but not as regards the illustrative figures of the original. Only 51 of the 110 figures of the original manuscript are reproduced and the artistic excellence of the originals has greatly suffered in the reproductions. Of the two facsimile pages, one (Plate vi) is rather imperfect as compared with the original,[1] while the distinctness of both has been impaired by reducing the size from 6″ by 8.5″ of the original to 3.5″ by 5.5″. A slight change in the form and size of the volume would have permitted a complete facsimile full size reproduction of the 26 pages of the manuscript, which would have greatly enhanced the value of the book. Notwithstanding, the volume

[1] The writer is indebted to his friend and colleague, Professor Louis Helmlinge of the Department of Romance Languages of this University, for the use of his facsimile reproduction of the original while writing this review.

will form an indispensable addition to the library of all who are interested in the history of the science of aviation.

R. E. MORITZ.

The Elements of Mechanics. By F. S. CAREY and J. PROUDMAN. London, Longmans, Green and Co., 1925. 314 pages. Price $3.00.

This book by a professor emeritus in the University of Liverpool and a young professor who has made notable advances in the study of the tides is a good example of what can be accomplished when the experience of a great teacher is combined with the brilliance of an ardent investigator. The subject is well presented, one good feature being the frequent use of vectors. Many illustrations are given to assist the reader and numerous examples are worked out at the end of the book to help along any student who is reading the subject for the first time.

Though the work is quite elementary, room is found for a useful chapter on frameworks and an elementary treatment of hydrostatics is included. We fully agree that instruction on these subjects should be given in a course on mechanics.

H. BATEMAN.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) title of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Isis, volume 7, no. 23, March, 1925: "Leibniz, the master builder of mathematical notations" by Florian Cajori, 412–429.

Mathematische Zeitschrift, volume 24, no. 3, December, 1925: "On the representation of functions by trigonometrical integrals" by N. Wiener, 575–616.

The Quarterly Journal of Pure and Applied Mathematics, volume 50, no. 3, October, 1925: "The general form of the suspension bridge catenary" by Ira Freeman, 269–271.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 2910 Harris Park Ave., Austin, Texas.

CLUB TOPICS

By B. H. BROWN, Dartmouth College

THEOREM OF BANG. ISOSCELES TETRAHEDRA

1. Let us state first two lemmas, the proofs of which are immediate.

Lemma 1. *If, between the 6 quantities a, b, c, a', b', c' there exist relations*

$$a+b+c=a+b'+c'=a'+b+c'=a'+b'+c\ ;$$

then

$$a=a'\ ;\qquad b=b'\ ;\qquad c=c'\ .$$

Lemma 2. *If, between the 4 positive quantities, A, B, C, D there exist relations*

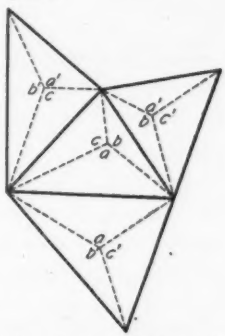
$$AB=CD\ ;\qquad AC=BD\ ;\qquad AD=BC\ ;$$

then

$$A=B=C=D\ .$$

Theorem of Bang.[1] *Let a sphere be inscribed in any tetrahedron. Let lines be drawn in the faces connecting the vertices with the points of tangency. The connecting lines in a face form 3 angles which are the same in each face.*

The accompanying figure shows the lines drawn in each face, the tetrahedron being opened out and developed on the plane. It is obvious that the small triangles which share a tetrahedral edge are congruent and the angles of the theorem may then be marked as in the figure. The proof is now complete by application of Lemma 1.



2. *Isosceles Tetrahedra.*[2] The word *isosceles* is applied, with some slight incongruity, to a tetrahedron whose opposite edges are equal, and consequently all of whose faces are congruent triangles.

Theorem. If the faces of a tetrahedron are isoperimetric, the tetrahedron is isosceles. The proof is immediate if we denote pairs of opposite edges by a, a'; b, b'; c, c' and employ Lemma 1.

---

[1] Proposed by Bang, *Tidskrift for Math.*, 1897, A, p. 48; proved by Gehrke, *ibid.*, p. 84. The extension of Bang's theorem to escribed spheres and to "attic-spheres" (Dachkugeln, sphères aux combles) has been made by Neuberg, *Jahresbericht*, 1907, p. 345. A dual theorem by Meyer, *Jahresbericht*, 1903, p. 137 may be stated as follows: "*If we circumscribe a sphere to a tetrahedron and at the vertices draw the tangent planes, and cut each tangent plane by the three tetrahedral planes meeting at that vertex, the traces of these last three (in the tangent plane) form three angles, which are the same at each vertex.*" Meyer's theorem is capable of an elegant elementary proof by stereographic projection and inversion. The theorems of Bang and of Meyer are metric specializations of a very general theorem on quadric surfaces proved by Meyer *l.c.*

A somewhat analogous theorem is due to White, *Bull. Am. Math. Soc.*, vol. 14, 1908, p. 220. Suppose a tetrahedron to be such that sums of opposite edges are equal, that is

$$a+a'=b+b'=c+c'.$$

It is then possible to find 4 numbers r_1, r_2, r_3, r_4 such that

$$\begin{aligned} r_1+r_2&=a, & r_3+r_4&=a',\\ r_2+r_3&=b, & r_4+r_1&=b',\\ r_3+r_1&=c, & r_2+r_4&=c'. \end{aligned}$$

Four spheres with radii r_1, r_2, r_3, r_4 and centers at the vertices of the tetrahedron will be mutually tangent. Four spheres have a common orthogonal sphere. This sphere will then be tangent to the *edges*

Theorem. If the faces of a tetrahedron are equiareal, the tetrahedron is isosceles. Proof: Let us represent the areas of the small triangles in the figure by the letters a, b, c; a', b', c'. Then, if the faces of the tetrahedron are equiareal $a=a'$, $b=b'$, $c=c'$ (Lemma 1). Let us denote the tangential distances from the vertices to the inscribed sphere by A, B, C, D. The equality of areas of small triangles gives 3 equations of which

$$\tfrac{1}{2}AB \sin b=\tfrac{1}{2}CD \sin b' ,$$

where b and b' now represent the Bang angles, is typical. Remembering that $b=b'$ etc. (Theorem of Bang), we have

$$AB=CD , \qquad AC=DB \qquad AD=BC ;$$

hence by Lemma 2, $A=B=C=D$, and the theorem follows readily. Incidentally it is now easy to establish the

Theorem. If the inscribed and circumscribed spheres of a tetrahedron are concentric, the tetrahedron is isosceles, and conversely.

We have now employed Lemma 1 in three ways, identifying the letters with (*a*) angles, (*b*) lengths of lines, (*c*) areas of triangles. Can this Lemma be employed in other connections? What further use can be made of Lemma 2? (Trigonometric functions of angles?) Are there analogous lemmas applicable to 4- or to n-dimensional varieties?

CLUB ACTIVITIES

THE JUNIOR MATHEMATICAL CLUB, University of Chicago, Chicago, Ill. [*1924*, 496]

The following papers were presented before the Junior Mathematical Club of the University of Chicago during the year 1924–1925:

October 22, 1924: "Constructions with one instrument only" by Miss Marion Stark.
November 5. "The mathematical library" by Dr. M. I. Logsdon.
November 19. "The postulational method in physics" by Mr. E. S. Akeley.
December 10. "Curve tracing by projective methods" by Mr. R. J. Garver.
January 22, 1925. "On satellites" by Professor K. Laves.
February 5. "Tchebycheff polynomials" by Dr. J. A. Shohat.
February 19. "Finite differences" by Mr. R. W. Barnard.
March 5. "Theories of infinity" by Mr. A. Blake.
April 22. "Semi-invariants of a ruled surface under affine transformations" by Mr. H. A. Simmons.
May 6. "Introduction to hyperbolic functions" by Mr. S. A. Rowland, Jr.

---

of the tetrahedron. For any such tetrahedron, if through each point of tangency and through the opposite edge there be passed planes, White has shown that "*These planes form three pairs which intersect the sphere in orthogonal pairs of circles; and that the right angles between these circles at any contact are bisected by the traces of two other planes, each containing four points of contact.*"

[2] The literature in connection with isosceles tetrahedra is extensive. As a starting point we mention: Genty, *Nouvelles Annales*, 1878, 2nd ser., vol. 17, p. 223; Lemoine, *Nouvelles Annales*, 1880, 2nd ser., vol. 19, p. 133; Chefik-Bey, *Nouvelles Annales*, 1880, 2nd ser., vol. 19, p. 403.

May 20. "Development of analytic geometry by means of line coordinates" by Mr. F. S. Nowlan.
June 3. "Affine differential geometry of ruled surfaces" by Mr. H. A. Simmons.

(Report by Mr. Sidney A. Rowland, Jr.)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OREGON, Eugene, Oregon.

[*1925*, 454]

During the year 1924–1925 the Mathematics Club of the University of Oregon was under the leadership of the following officers:

President.....Marie Ridings
Vice-president....Sylvia Veatch
Secretary.....Eula Benson
Treasurer.....Roland Humphreys.

The following programs were given:
October 1924. "Magic squares" by Marie Ridings.
November. "History of π" by Sylvia Veatch.
December. "Trisection of an angle, squaring a circle, and doubling a cube" by Roland Humphreys.
January 1925. "Non-euclidean geometry" by Vladimir Rojonsky.
February. "Japanese mathematics" by Eula Benson. "Women in science" by Caroline Felton.
April. "The theory of the gyroscope" by Walter Brattain.
May. Annual picnic and initiation of new members.
June. "The quantum theory" by Leonard Neuman.
The officers elected to serve for the year 1925–1926 are as follows:

President.....Eula Benson
Vice-president....Evan Lapham
Secretary.....Helen White
Treasurer.....Herbert Yearian.

(Report by Miss Eula Benson)

THE GRINNELL COLLEGE MATHEMATICS CLUB, Grinnell, Iowa

[*1925*, 91]

The officers of the Grinnell College Mathematics Club for the year 1924–1925 were:

President.....Margaret Field
Vice-president.....Ben Morgart
Secretary-treasurer.....Ada Grosenbaugh.

The average attendance at the meetings was thirteen. The Club was particularly fortunate in having with them Professor George D. Birkhoff, Harvard Exchange Professor for 1924. The Club endeavored to study at each meeting, one of the constellations prominent at that time.

The following programs were presented:
"Life of Professor Birkhoff" by Dirk Heezon.
"The first works in mathematics in the New World" by Ruth Dougherty.
"The constellation Lyra" by Edith Wier.
"Mathematical recreations" by Mary Pilgrim.
"The constellation Pegasus" by Eugene Woodruff.
"Sexagesimal fractions" by Lela Kaisand.
"The four color problem" by Professor Birkhoff.
"The life of Leibniz" by Harriet Allen.
"The constellation Orion" by Alice Clifton.
"Mathematical fallacies" by Clarke Morris.
"The constellations Cygnus and Pleiades" by Ada Grosenbaugh.
"Rabbi Ben Ezra" by Harold Sweeney.
"Recreational mathematics" by Maude Martens.
"The theory of numbers" by Professor Rusk.

"The life of Archimedes" by Ben Morgart.
"The constellation Gemini" by Barbara Davidson.
"Relation of mathematics to another science, that of shape and form" by Dorothy Wilson.
"Report on the study of high school and college mathematics grades" by Howard Smith.
"Big numbers and little numbers" by Mary Wilson.
"Elementary applications of the theory of probabilities" by Professor McClenon.
"Curves of life" by Ida Iverson.
Summary of the year's work by Margaret Field.

(Report by Miss Ruth Dougherty)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3173. Proposed by A. A. Bennett, Lehigh University.

Let n black balls and n white balls be placed together in an urn. Let these balls be then withdrawn one at a time, none being replaced. Let $P(m)$ denote the probability of there occurring at least one sequence of m consecutive white balls during the complete process of withdrawing all the balls. Let m' denote the maximum value of m for which $P(m) \geq \frac{1}{2}$. Determine m' as a function of n with reference also to its asymptotic character. Generalize to the case of n balls each of more than two colors.

3174. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Construct a triangle given an angle, the sum of the including sides, and the distance of the vertex of the given angle from the orthocenter of the triangle.

NOTE BY THE EDITORS: Consider the case where a linear integral function of the including sides is given.

3175. Proposed by J. B. Reynolds, Lehigh University.

Prove that if the perpendiculars from each of three vertices of a tetrahedron upon the opposite faces meet in a point, the fourth will also pass through that point.

3176. Proposed by J. B. Reynolds, Lehigh University.

Through the vertex, O, of a tetrahedron, $OABC$, are passed three planes perpendicular respectively to OA, OB, and OC. Let BC (extended if necessary) cut the first plane in F, CA cut the second in G, and AB cut the third in H. Prove FGH a straight line.

3177. Proposed by J. L. Riley, Ouachita College, Arkadelphia, Ark.

An elliptic wall having a major axis $2a$ and a minor axis $2b$ stands in the middle of a large field. A horse is tethered to the outside of this wall at the extremity of the minor axis by a rope the length of which is equal to half the periphery of the wall. Find the area of the ground grazed over.

3178. Proposed by B. C. Wong, Berkeley, California.

On OP, the radius vector of any point on the cardioid $\rho=2a(1-\cos\theta)$, as diagonal a rectangle $OMPN$ is constructed with the sides OM and ON bisecting the angles between the radius vector and the axis of the cardioid. Find the loci of M and N and give their constructions without the use of the cardioid.

3179. Proposed by Einar Hille, Princeton University.

The differential equation $y'=y^{2/3}$ has two independent solutions which pass through the origin, namely $y\equiv 0$ and $y=\frac{1}{27}x^3$. Prove that these solutions can be obtained by the method of successive approximations using as first approximation in the former case $y_0=0$, in the latter case $y_0=x^{a_0}$ where a_0 is any positive constant.

3180. Proposed by J. Rosenbaum, Milford, Connecticut.

Given the angles of an inscribed polygon and the radius of circumscribed circle to construct the polygon.

SOLUTIONS

2803. [1920, 31]. Proposed by S. A. Corey, Des Moines, Iowa.

In the November, 1918, number of the *Proceedings of the Edinburgh Mathematical Society* (Vol. 36, part 2, page 103), Professor Whittaker gives the following formula for the solution of algebraic and transcendental equations:

The root of the equation

$$0=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots,$$

which is smallest in absolute value, is given by the series

$$-\frac{a_0}{a_1}-\frac{a_0^2a_2}{a_1\begin{vmatrix}a_1a_2\\a_0a_1\end{vmatrix}}-\frac{a_0^3\begin{vmatrix}a_2a_3\\a_1a_2\end{vmatrix}}{\begin{vmatrix}a_1a_2\\a_0a_1\end{vmatrix}\begin{vmatrix}a_1a_2a_3\\a_0a_1a_2\\0\ a_0a_1\end{vmatrix}}-\frac{a_0^4\begin{vmatrix}a_2a_3a_4\\a_1a_2a_3\\a_0a_1a_2\end{vmatrix}}{\begin{vmatrix}a_1a_2\\a_0a_1\end{vmatrix}\begin{vmatrix}a_1a_2a_3\\a_0a_1a_2\\0\ a_0a_1\end{vmatrix}\begin{vmatrix}a_1a_2a_3a_4\\a_0a_1a_2a_3\\0\ a_0a_1a_2\\0\,0\ a_0a_1\end{vmatrix}}-\cdots.$$

In case of any algebraic equation with imaginary or complex roots the above formula clearly fails. State the conditions under which the formula may be relied upon to give correct results.

Solution and Discussion by Otto Dunkel, Washington University.

The statement of the theorem in this problem is incorrect and a revised statement will be given. The series, when written correctly, is merely a modification of a known process for approximating the roots of an algebraic equation (*1925*, 354–370) and it is not so convenient for computation as this process.

The equation

$$a_0+a_1x+a_2x^2+\cdots+a_rx^r=0\,,\qquad a_0a_r\neq 0\,. \tag{1}$$

will be considered where the coefficients a_i may be real or complex. In the denominators of terms of the series occur determinants of order n of the type

$$D(n)=\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & & & \\ a_0 & a_1 & a_2 & \cdots & & & \\ 0 & a_0 & a_1 & \cdots & & & \\ 0 & 0 & a_0 & \cdots & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ 0 & 0 & 0 & \cdots & 0 & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_0 & a_1 \end{vmatrix}, \tag{2}$$

which when developed in terms of the elements of the first row give the recurrent relation

$$D(n)=a_1D(n-1)-a_2a_0D(n-2)+\cdots+(-1)^{r-1}a_ra_0^{r-1}D(n-r)\,, \tag{3}$$
$$D(n)=0\,, \qquad -r+1\leqq n\leqq -1\,, \qquad D(0)=1\,, \qquad D(1)=a_1\,.$$

If we set

$$P(n+r)=(-1)^n\frac{D(n)}{a_0^n}\,, \tag{4}$$

this may be written in the simpler form

$$a_0P(n+r)+a_1P(n+r-1)+a_2P(n+r-2)+\cdots+a_rP(n)=0\,. \tag{5}$$

Then

$$\operatorname*{Limit}_{n=\infty}\frac{P(n)}{P(n+1)}=\frac{1}{c_1}=d_1\,, \tag{6}$$

if the equation

$$a_0y^r+a_1y^{r-1}+a_2y^{r-2}+\cdots+a_r=0 \tag{1'}$$

has a root c_1 whose absolute value is greater than that of any other root, i.e. d_1 is the root of smallest absolute value of the equation (1).[1] The conditions for the existence of the limit (6) will be stated more precisely later. It will now be shown that the series of the problem, when it has a meaning, is merely a modification of (6).

The cofactors of $D(n)$ corresponding to the elements at the extremities of the diagonals are $D(n-1)$, $D(n-1)$, $(-1)^{n-1}a_0^{n-1}$ and $(-1)^{n-1}E(n-1)$, where

$$E(n)=\begin{vmatrix} a_2 & a_3 & \cdots & & & & \\ a_1 & a_2 & \cdots & & & & \\ a_0 & a_1 & \cdots & & & & \\ 0 & a_0 & \cdots & & & & \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & & & & & \\ 0 & 0 & \cdots & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & \cdots & 0 & a_0 & a_1 & a_2 \end{vmatrix}\,. \tag{7}$$

is a determinant of order n. Hence[2]

$$D(n-2)D(n)=D^2(n-1)-a_0^{n-1}E(n-1)\,. \tag{8}$$

Suppose now that $D(p-1)=0$, $D(n)\neq 0$, $n\geqq p$, then

$$\frac{D(n-1)}{D(n)}-\frac{D(n-2)}{D(n-1)}=\frac{a_0^{n-1}E(n-1)}{D(n-1)D(n)}\,, \qquad n\geqq p+1\,.$$

Summing these relations and multiplying the result by $-a_0$ we have

$$\frac{P(n)}{P(n+1)}=-\frac{a_0D(n-r)}{D(n-r+1)}=-\sum_{m=p}^{m=n-r}\frac{a_0^{m+1}E(m)}{D(m)D(m+1)}\,. \tag{9}$$

This completes the proof. The series given in the problem can be used only if $p=0$ and the limit (6) exists. An example will be given later where the limit exists and for which p is not zero.

The vanishing of the determinants $D(n)$ indicates in general the existence of imaginary roots in the case where the coefficients a_i are all real. Suppose, for example, $D(p)\neq 0$, $D(p+q+1)\neq 0$, $D(n)=0$, $p+1\leqq n\leqq p+q$, $q<r$. We cannot have $q\geqq r$, for then by (3) all of the $D(n)$'s would be zero contrary to the fact that $D(0)=1$. We have then the following cases:

---

[1] See the article cited above.

[2] See Bôcher's *Higher Algebra*, pp. 31-33.

Case I. p odd.

If q is odd there are at least $q+1$ imaginary roots.

If q is even there are at least q or $q+2$ imaginary roots according as $D(p+q+1)$ is greater or less than zero.

Case II. p even.

If q is even there are at least q or $q+2$ imaginary roots according as $D(p)$ is greater or less than zero.

If q is odd and $D(p)D(p+q+1)<0$ there are at least $q+1$ imaginary roots; if $D(p)>0$ and $D(p+q+1)>0$ there are at least $q-1$; if $D(p)<0$ and $D(p+q+1)<0$ there are at least $q+3$.

The proofs of these statements follow easily by methods similar to those used by the writer in the paper[1] "Necessary Conditions for the Reality of all the Roots of an Algebraic Equation."

It is obvious that series similar to (9) may be written for the root of largest absolute value. If the roots have unequal absolute values and $|c_1|>|c_2|>|c_3|>\cdots$, series of a more complicated form may also be written for the products c_1c_2, $c_1c_2c_3$, $c_1c_2c_3c_4$, $\cdots$. If two roots have equal absolute values and the same multiplicities, say $|c_1|>|c_2|>|c_3|=|c_4|>|c_5|>\cdots$, then series may be written for c_1, c_1c_2, $c_1c_2c_3c_4$. If, however, the multiplicity of c_3 is greater than that of c_4 then series may be written for $c_1c_2c_3$ and $c_1c_2c_3c_4$. These series are not as convenient for computation as their equivalents, that is the ratio $P(n+1)/P(n)$ for c_1 and the ratios of certain determinants of the $P(n)$'s in the other cases. The proofs of these statements, as well as the following, are rather long and will not be given here, since the writer expects to present them in a paper which will appear soon. We shall now state the conditions that the limit (6) shall exist or that the infinite series obtained from (9) shall converge.

If $c_1, c_2, c_3, \cdots, c_m$ are distinct roots of (1′) whose absolute values are equal and this absolute value is greater than that of any other remaining root, if any, and if further the multiplicity of c_1 exceeds that of c_2, of $c_3, \cdots$, of c_m, then

$$\operatorname*{Limit}_{n=\infty}\frac{P(n+1)}{P(n)}=c_1\,.$$

In no other case does a limit exist. When these conditions for a limit are satisfied, and if $D(n)=0$ for some values of n, then it may be shown that for some value of n, say $n=p$, $D(p-1)=0$, $D(n)\neq 0$, $n\geqq p$.

If the coefficients a_i are all real then c_1 must be real in order to satisfy the above requirements for a limit. There can be only one other real root in the set of m, say c_2 where $c_2=-c_1$.

An example. As an example for which some of the $D(n)$'s are zero we shall take the equation

$$1-x+x^2-x^3-x^4=0\,, \tag{10}$$

which is a slight modification of a special case of the equations in Problem **3158**, (*1926*, 47). From the theorem of that problem it will be seen that the absolute values of the two imaginary roots lie between those of the two real roots. Either one of the two real roots may then be computed by the series, or, more simply, from the quotient of two successive values of $P(n)$. The difference equation in one case is

$$P(n+4)-P(n+3)+P(n+2)-P(n+1)-P(n)=0\,, \tag{11}$$
$$P(4)=1\,,\qquad P(n)=0\,,\qquad 1\leqq n\leqq 3\,.$$

Computing a few values we have the adjoined table

n	$P(n)$	n	$P(n)$
1	0	9	3
2	0	10	1
3	0	11	0
4	1	12	4
5	1	13	8
6	0	14	5
7	0	15	1
8	2	16	8

It follows from (4) that $D(2)=D(3)=D(7)=0$. We may use in place of (11) the equation

$$P(n+5)=2P(n+1)+P(n),$$

if we adjoin to the initial values of (11) $P(5)=1$. This simpler equation shows that $P(n)\neq 0$ for $n\geqq 12$ or $D(n)\neq 0$ for $n\geqq 8$. Hence in the series (9) for this example $p=8$, and the series will surely converge to the real root of smaller absolute value, but as already stated it would be much simpler to continue this table. The convergence is very slow so that $P(200)/P(201)$ would give an approximation which is correct in only

[1] *Washington University Studies*, vol. 6. Scientific Series, no. 1, pp. 13-19, 1918.

about the first four decimals. To illustrate the computation of all the roots we shall consider the equation

$$x^4-x^3+x^2-x-1=0\,, \tag{10'}$$

whose roots are the reciprocals of the roots of (10). In order to improve the convergence in the case of the larger root we shall reduce the roots by 1.3 and then change their signs. This gives the equation

$$.0491P(n+4)-5.318P(n+3)+7.24P(n+2)-4.2P(n+1)+P(n)=0\,.$$

From this equation we find the successive values of $P(n)$; 0, 0, 0, 1, 108.31, 11583.60, 1238734.513, 132468000.871. Then one root is approximately

$$1.3-\frac{P(7)}{P(8)}=1.3-.00935119803=1.29064880197\,.$$

This is correct up to and including the 8th decimal. For the second real root, after changing its sign, we have the equation

$$P(n+4)=P(n+3)+P(n+2)+P(n+1)+P(n)\,.$$

Here the convergence is more rapid and the computation of the root is fairly easy.

n	$P(n)$	n	$P(n)$	n	$P(n)$	n	$P(n)$
1	0	10	29	19	10671	28	3919944
2	0	11	56	20	20569	29	7555935
3	0	12	108	21	39648	30	14564533
4	1	13	208	22	76424	31	28074040
5	1	14	401	23	147312	32	54114452
6	2	15	773	24	283953	33	104308960
7	4	16	1490	25	547337	34	201061985
8	8	17	2872	26	1055026		
9	15	18	5536	27	2033628		

From the adjoined table we have as an approximation to this root

$$\frac{P(33)}{P(34)}=.51879006367116$$

in which the last three digits are incorrect. The negative root of (10′) is then $-.5187900637$.

It is also possible to approximate the imaginary roots in the same way. Reducing the roots of (10′) by i in order to make one of the imaginary have a smaller absolute value than that of any other root we obtain the equation

$$P(n+4)=(2-2i)P(n+3)-(5+3i)P(n+2)+(-1+4i)P(n+1)+P(n).$$

n	$P(n)$	n	$P(n)$
1	0	7	$-49-4i$
2	0	8	$-107+160i$
3	0	9	$390+690i$
4	1	10	$3235-82i$
5	$2-2i$	11	$5844-11846i$
6	$-5-11i$	12	$-31682-43645i$

This gives the adjoined table, and from it we have

$$\frac{P(11)}{P(12)}=.114098+.216722i\,.$$

Hence the imaginary roots are approximately $.114098\pm1.216722i$.

A calculation by a second method gave $.114071\pm1.216746i$. This last calculation is probably correct in the figures given. The absolute value of either imaginary root is 1.2220814, and, since this differs very little from the positive root, we see why the original of sequence values for $P(n)$ gave such a slow convergence.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

At Denison University, Professor F. B. WILEY has resumed his work as head of the department of mathematics, after a year's leave of absence spent as exchange professor at Robert College, Constantinople. Professor E. C. RUPP, who was head of the department of mathematics of Shurtleff College last year, has been appointed assistant professor. Mr. DONALD FITCH has been promoted from part-time instructor to full-time instructor.

The book entitled "Algebras and Their Arithmetics" by Professor L. E. DICKSON, which was published two years ago by the University of Chicago Press, is now being translated into German and will soon be published at Zurich, Switzerland.

Professor ARNOLD DRESDEN, University of Wisconsin, was the symposium speaker at the recent meetings of the American Mathematical Society held at the University of Chicago.

On February 1, 1926, an organization meeting was held of the Trustees of the National Research Endowment, recently established by the National Academy of Sciences; Secretary Herbert Hoover was chairman. On this occasion a declaration was made public reciting the purpose of this Endowment, which is to provide adequate funds for research in pure science in this country. Professor OSWALD VEBLEN is a member of the Board of Trustees.

The following appointments to Guggenheim Fellowships are announced: Professor E. P. LANE, of the University of Chicago, for a comparison of American and Italian methods in projective differential geometry; Professor E. B. STOUFFER, of the University of Kansas, for research in projective differential geometry; Professor NORBERT WIENER, of the Massachusetts Institute of Technology, for the study of certain topics forming extensions of the theories of Fourier series and Fourier integrals.

At Columbia University, Professor JULES DRACH, of the chair of analysis applied to geometry at the University of Paris, has been appointed visiting professor for the academic year 1926–27; Professor Drach will lecture in English.

At Stanford University, Professor R. L. GREEN has been made head of a new department of mathematics formed by the combination of the mathematics department and part of the department of applied mathematics; some of the members of the department of applied mathematics have been transferred to the new school of engineering.

The Nobel prize in Physics for 1924 has been awarded to Professor K. M. G. Siegbahn of the University of Upsala for his work in spectrum analysis.

The following additional reports of Summer Sessions to be held in 1926 have been received:

Stanford University. By Dr. KELLEY; Differential equations; By Dr. HAROLD HOTELLING; Theory of functions of a complex variable; By Professors H. F. BLICHFELDT and W. A. MANNING; Theory of groups, Calculus of variations.

Bucknell University, July 6–August 13. In addition to courses in College algebra and Calculus the following advanced courses are given: by Professor H. S. EVERETT: Teaching of junior and senior high school mathematics, Mathematical theory of interest and insurance (second course), Introduction to mathematical philosophy. By Assistant Professor GOLD; Synthetic projective geometry.

The following 27 doctorates with mathematics or mathematical physics as major subject were conferred by American universities during 1925; the university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case, if available.

ETHEL L. ANDERTON, Yale, June, Bioche curve pairs.

CLIFFORD BELL, California, December, chemistry, The triangles in and circumscribed to the quartic curves of deficiency zero.

T. L. BENNETT, Illinois, July (date of examination), physics, Mapping by means of linear systems of curves invariant under Cremona involutions.

H. W. BRINKMANN, Harvard, February, Contributions to the theory of Riemann spaces.

P. A. CARIS, Pennsylvania, June, A solution of the quadratic congruence, Modulo p, $p=8n+1$, n odd.

M. G. CARMAN, Illinois, May, theoretical physics and theoretical and applied mechanics, Expansion problems in connection with homogeneous linear q-difference equations.

E. F. COX, Cornell, September, physics, The polynomial solutions of the difference equation $af(x+1)+bf(x)=\varphi(x)$.

H. M. GEHMAN, Pennsylvania, February, Concerning the subsets of plane continuous curves.

V. G. GROVE, Chicago, September, A theory of a general net on a surface.

R. W. HARTLEY, Pennsylvania, June, Determination of the ternary collineation groups whose coefficients lie in the $NGF(2^n)$.

D. L. HOLL, Chicago, September, Viscous fluid motion in eccentric cylinders.

J. W. HURST, Illinois, August, physics and astronomy, The classification of ternary algebras.

SAMUEL JACOBSON, Yale, June, mathematical physics, Temperature radiation and thermal equilibrium.

ANNA M. LEHR, Bryn Mawr, June, physics, The plane quintic with five cusps.

R. G. LUBBEN, Texas, June, philosophy, The double-elliptic case of the Lie-Riemann-Helmholtz-Hilbert problem of the foundations of geometry.

A. J. MARIA, Rice Institute, June, Functions of plurisegments.

R. M. MATHEWS, Illinois, April, physics, Cubic curves and desmic surfaces.

L. T. MOORE, Johns Hopkins, June, zoology, Determination of the type of a tricrunodal quartic from its invariants.

F. S. NOWLAN, Chicago, September, Arithmetics of rational division algebras of order nine.

ECHO D. PEPPER, Chicago, September, Theory of algebras over a quasi-field.

P. G. ROBINSON, Chicago, June, Surfaces with constant absolute invariants.

MEYER SALKOVER, Yale, June, mathematical physics, Bateman's extended electrodynamics and the mass of an electron.

H. A. SIMMONS, Chicago, June, The first and second variations of a double integral for the case of variable limits.

MARCUS SKARSTEDT, California, May, physics, The normal quartic curve of four-space.

V. A. TAN, Chicago, June, Projective properties of the Moutard quadric.

TAKASHI TERAMI, California, December, physics, The solution of the differential equation of a vibrating membrane by successive approximations.

L. E. WARD, Harvard, June, Third order boundary value problems and the allied expansions.

The following doctorate with education as major and mathematics as minor subject was conferred:

LAO G. SIMONS, Columbia, February, Introduction of algebra into American schools in the eighteenth century.

The Mathematical Association was represented by Professor ETHELWYNN R. BECKWITH at the diamond jubilee of Milwaukee-Downer College on June 12, by Professor E. C. KIEFER at the twenty-fifth anniversary of the founding of the James Millikin University, April 29 to May 1, and by Professor H. B.

EVANS at the inauguration of Charles Ezra Beury as president of Temple University on May 7.

Professor H. W. KUHN has been appointed to the chairmanship of the department of mathematics of Ohio State University, following the resignation of Professor R. D. BOHANNAN.

It is announced that Professors E. T. BELL, of the University of Washington, and ANNA PELL-WHEELER, of Bryn Mawr College, have accepted the invitations of the Council of the American Mathematical Society to give Colloquia at the 1927 Summer Meeting in Madison, Wisconsin, their topics being respectively "Algebraic arithmetic" and "The theory of quadratic forms in infinitely many variables, and applications." The 1927 Summer Meeting of the Mathematical Association of America will be held at Madison in connection with that of the Society.

The sixth International Congress of Philoscphy will meet September 13–17, 1926, at Harvard University. In each of four divisions, Metaphysics, Logic and Philosophy of Science, Theory of Values, and History of Philosophy, there will be one general session and four section meetings.

Professor M. I. PUPIN, of Columbia University, has received the honorary degree of doctor of laws fr[illegible] the University of California.

Assistant Professor N[illegible]M. ALDERTON, of Mills College, has been promoted to an associate professor[illegible] of mathematics.

Mr. W. C. ARNOLD, of DePauw University, has been promoted to an assistant professorship of mathematics.

Associate Professor W. H. BERRY has been promoted to a full professorship of mathematics at the Brooklyn Polytechnic Institute.

Mr. R. M. CHASE has been appointed professor of civil engineering and mathematics at the Alaska Agricultural College and School of Mines, Fairbanks.

Associate Professor T. W. FRAVATT, of Pennsylvania State College, has been promoted to a full professorship of mathematics.

Assistant Professor V. G. GROVE, of Michigan State College, has been promoted to an associate professorship of mathematics.

Dr. GOLDIE P. HORTON has been promoted to an adjunct professorship of promoted to an associate professorship of mathematics.

Dr. GOLDIE P. HORTON has been promoted to an adjunct professorship of mathematics at the University of Texas.

Associate Professor EMILIE N. MARTIN, of Mount Holyoke College, has been promoted to a full professorship of mathematics.

Dr. R. M. MATHEWS, of the University of Illinois, has been appointed associate professor of mathematics at the University of West Virginia.

Assistant Professor T. A. PIERCE, of the University of Nebraska, has been promoted to an associate professorship of mathematics.

Dr. R. G. PUTNAM has been promoted to an assistant professorship of mathematics at New York University.

Associate Professor W. P. RUSSELL has been promoted to a full professorship of mathematics at Pomona College.

Assistant Professor J. H. TAYLOR, of Lehigh University, has been appointed assistant professor of mathematics at the University of Wisconsin.

Assistant Professor W. L. G. WILLIAMS, of McGill University, has been promoted to an associate professorship of mathematics.

Assistant Professor C. G. YEATON has been promoted to an associate professorship of mathematics at Oberlin College.

Assistant Professor W. C. GRAUSTEIN of Harvard University has been promoted to an associate professorship of mathematics.

Associate Professor A. F. CARPENTER of the University of Washington has been promoted to a full professorship of mathematics.

Associate Professor L. L. SMAIL, of the U[illegible]ity of Texas, has been appointed to an associate professorship of math[illegible]cs at Lehigh University.

Professor R. A. JOHNSON of Hamlin University has been appointed assistant professor of mathematics at Hunter College.

The following appointsments to instructorships are announced:

Harvard University, Mr. H. B. CURRY, Mr. E. H. CULTER, Mr. H. H. HINRICHSEN, Mr. W. A. JENKINS, Mr. F. C. JONAH, Mr. MORRIS MARDEN, Mr. ROBIN ROBINSON, Mr. GEORGE SANTE, Mr. T. D. SMITH, and Mr. S. B. SOMMERVILLE.

Hunter College, Miss LAURA GUGGENBUHL, Miss MARGARET A. YOUNG, Miss LYLAH KRYDER, and Miss MARGARET LITTANER.

At Rutgers University an *Institute of Mathematics* has been organized as a part of the summer session to be held from June 28 to August 6, 1926. The immediate work of the institute will be in charge of Associate Professor F. D. MURNAGHAM of Johns Hopkins University. There will be associated with him as members of the staff and special lecturers the following: Professor L. P. EISENHART and Mr. ALONZO CHURCH of Princeton University, Professor E. R. HEDRICK of the University of California, Professor H. H. MITCHELL of the University of Pennsylvania, Professor RICHARD MORRIS of Rutgers University, and Professor L. D. HAERTTER of University High School, University of

Minnesota. The advanced courses to be given are as follows: By Professor Murnaghan: Functions of a Complex Variable, Modern Higher Algebra. By Professor Murnaghan and other members of the staff: Topics and Modern Theories in Mathematics, Seminar and Conference Period. The following courses, designed especially for secondary school teachers, are also offered in the Institute: By Mr. Church, Topics in Coordinate Geometry and Calculus; Modern Elementary Geometry. By Professor Haertter, The Teaching of Mathematics in the Junior High School; The Teaching of Mathematics in the Senior High School; The History of Mathematics. Besides the courses in the institute, the usual elementary undergraduate courses in geometry, college algebra, analytical geometry, and the calculus are offered in the regular summer session.

Assistant Professor WALTER DENSTON of Kenyon College was shot on April 8th while out alone shooting at targets. He was a graduate of Christ's College, Cambridge, and a teacher in the Russian Imperial Naval Academy at Kronstadt and in Ashbury College, Ottawa, Canada, before coming to the United States.

Professor F. N. COLE, of Columbia University, for twenty-five years secretary of the American Mathematical Society and editor of the *Bulletin*, died on May 26, 1926, at the age of sixty-four. Death was caused by heart failure brought on by an infected tooth. He was to have retired from teaching on September 20, his next birthday.

Cambridge University Press
(England)

MATRICES AND DETERMINOIDS

By C. E. Cullis

This voluminous treatise by the Professor Emeritus of Mathematics, University of Calcutta, has been brought nearer completion by the publication of Volume III, part 1. The work is an amplification of a course of lectures given for the University of Calcutta. Its chief feature is that it deals with rectangular matrices and determinoids as distinguished from square matrices and determinants. An attempt has been made to set forth a complete and consistent theory or calculus of rectangular matrices and determinoids.

The first volume contains the most fundamental portions of the theory and concludes with a solution of any system of linear algebraic equations.

The second volume elaborates those parts of the theory which precede the investigation of the functional properties of matrices. It deals almost exclusively with matrices whose elements are constants and with those transformations of matrices which are classed as equigradent.

Application of these theories is made in the third volume, part 1.

"In a reference library, the volumes will be indispensable on account of their wealth of elementary formulae."—*The Mathematical Gazette.*

Prices: Vol. I—$8.00; Vol. II—$14.00; Vol. III—$21.00

Agents in the United States

THE MACMILLAN COMPANY

60 Fifth Avenue **NEW YORK CITY**

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of.....Dollars, to be known as the.....Fund, and to be used for[2] { Endowment—the income only of which may be expended. / Special Projects—for which both principal and income may be expended. }

Witness:..... Signature.....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

CONTENTS

Tenth Annual Meeting of the Association. By W. D. CAIRNS 169
A Proposed Amendment to the By-Laws. By W. D. CAIRNS 181
May (1925) Meeting of the Minnesota Section. By A. L. UNDERHILL 183
The Course in Statistics in the Mathematics Department. By A. R. CRATHORNE 185
The Sturm and Fourier-Budan Theorems and Mixed Differential-difference Equations. By TOMLINSON FORT 194
A Cross-division Process and its Application to the Extraction of Roots. By By D. H. LEHMER 198
Commutative Algebraic Inversions. By E. T. BELL 206
QUESTIONS AND DISCUSSIONS:
Discussions—"A cubic equation of Newton's" by NORMAN ANNING: "Another poristic system of triangles" by RUFUS CRANE: "A new method of determining Bernoulli's numbers" by C. A. MESSICK: "A crystallographic illustration of quadratic involution" by A. C. LUNN: "On the duals of metric theorems" by LOUIS WEISNER: "A note on the Hindu-Arabic numerals" by BIBHUTIBHUSAN DATTA 210
RECENT PUBLICATIONS: Reviews by R. E. MORITZ, H. BATEMAN.
Articles in current periodicals 222
UNDERGRADUATE MATHEMATICS CLUBS: Club topics—"Theorem of Bang, Isosceles tetrahedra" by B. H. BROWN. Club activities—University of Chicago, University of Oregon, Grinnell College 224
PROBLEMS AND SOLUTIONS: Problems for solution—3173-3180.
Solution of 2803 by OTTO DUNKEL 228
NOTES AND NEWS 233

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Tenth Summer Meeting of the Association, Columbus, Ohio, September 7-8, 1926.

Eleventh Annual Meeting, Philadelphia, Pa., December, 30-31, 1926.

The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.

INDIANA, Purdue University, May, 7-8.

IOWA, Cedar Rapids, April.

KANSAS, Merged in National Meeting.

KENTUCKY, Berea College, May 1.

LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., December 4.

MICHIGAN, Ann Arbor, Mich., April 1.

MINNESOTA, Northfield, Minn., May 22.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Bethany, Neb., May.

OHIO, Columbus, Ohio, April 2.

ROCKY MOUNTAIN, Colorado College, April, 1927.

SOUTHEASTERN, Atlanta, Ga., March 19-20.

SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.

TEXAS, November.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

169
181
183
185
194
198
206
210
222
224
228
233

CHIEF.

N. Y.

SURER

.

er.

April,

19-20.

Calif.,

tor.

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The fourth annual meeting of the Southeastern Section of the Mathematical Association of America was held at the Philips High School, Birmingham, Alabama, March 20-21, 1925. Friday night a special dinner was given in honor of Professors Oswald Veblen and Archibald Henderson with Professor T. R. Eagles presiding as toastmaster. About forty were present for the dinner and the other sessions of the meeting including the following sixteen members of the Association: S. M. Barton,[1] J. B. Coleman, O. A. Culmer, T. R. Eagles, L. Farley, A. Henderson, C. D. Killebrew, Katherine Meckle, A. B. Morton, Frank Ordway, W. P. Ott, M. T. Peed,[2] W. W. Rankin Jr., D. Rumble, Oswald Veblen.

The following officers were elected for 1925–1926; Chairman, M. T. PEED, Emory University; Vice-chairman, W. P. OTT, University of Alabama; Secretary-treasurer, W. W. RANKIN, Jr., Agnes Scott. Professors A. B. Morton and T. R. Eagles were appointed to act with the secretary-treasurer as a Program Committee. A special committee was appointed to make a study of "Closer correlation of high school and college mathematics" W. W. Rankin, Jr., Chairman; T. R. Eagles, Frank Ordway, W. P. Ott, D. Rumble.

The following papers were presented: (1) "The universe; finite or infinite?" by Professor ARCHIBALD HENDERSON; (2) "Geometry as a branch of physics," by Professor OSWALD VEBLEN; (3) "Projective geometry in the college curriculum," by Professor M. T. PEED; (4) "The practical side of pure mathematics," by Professor S. M. BARTON; (5) "Notations for determinants," by Professor OSWALD VEBLEN; (6) "Honor courses at Cambridge university," (by title) by Professor R. P. STEPHENS; and (7) "A new mode of approach to relativity," by Professor ARCHIBALD HENDERSON. Discussion; "Connecting high school and college mathematics" led by Professor T. R. EAGLES.

Abstracts of the papers are given below, the numbers corresponding to the numbers in the list of titles as above.

1. In this paper were presented the results of some studies, ranging over a period of years, in this fascinating subject. Dr. Henderson recently had the advantage of discussing the subject with Professor Albert Einstein at the University of Berlin. To Professor Einstein is due the notion of the finite, but unbounded universe, which is a logical consequence of his general relativity theory. This paper, which had recently been given by radio under the auspices of the

[1] Professor S. M. Barton died January 5, 1926.

[2] Professor M. T. Peed died August 28, 1925.